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THE ELEMENTS OF THE CONIC SECTIONS,

ADAPTED TO THE

USE of STUDENTS in PHILOSOPHY.

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BY THE

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THE ELEMENTS OF THE CONIC SECTIONS.

ON THE PARABOLA.

DEFINITIONS.

FIG. 1.

1. LET a point *S* be aſſumed in the line *LZ*, and *ELD* be drawn perpendicular to it; then if a line *MW* move parallel to *LZ*, interſecting another line *SH* revolving about *S*, ſo that *MP* may be always equal to *SP*, the curve paſſing through all the interſections *P* is called a *Parabola*.

COR. Hence, the curve cuts *LZ* in a point *A* which biſects *LS*, and *A* is called the *Vertex*. Alſo, the two parts of the curve on each ſide of *AZ* are ſimilar and equal.

2. The point *S* is called the *Focus*.

3. The line *AZ* is called the *Axis*; and any line *PW* parallel to *AZ*, is called a *Diameter*.

4. The line *ED* is called the *Directrix*.

5. If PN be drawn perpendicular to AZ, AN is called an *Abſciſſa*, and PN an *Ordinate* to the axis.

6. If QV be drawn parallel to a tangent PT at P, PV is called an *Abſciſſa*, and QV an *Ordinate* to the diameter PW. Alſo, TN is called the *Sub-tangent*.

7. Four times the diſtance PS, is called the *Latus Rectum*, or *Parameter*, to the diameter PW. And the latus rectum to the axis is called the *Principal* latus rectum.

8. If PG be perpendicular to the curve, it is called the *Normal*; and NG the *Sub-normal*.

Prop. I.

The diſtance of the focus from any point of the curve, is equal to the ſum of the abſciſſa belonging to that point, and the diſtance of the vertex from the focus.

For (Def. 1.) $SP = PM =$ (as $MPNL$ is a parallelogram by conſtr.) $NL = AN + AL = AN + AS$ (Def. 1. Cor.)

Prop. II.

The diſtance of the vertex from the focus, is equal to one fourth of the principal latus rectum.

Draw CSB perpendicular to AZ; then when P comes to B, N coincides with S, and (Prop. 1.) $SB = SA + SA = 2SA$; therefore $BC =$ (Def. 1. Cor.) $2SB = 4SA$ the latus rectum (L) to the axis, or the *principal* latus rectum (Def. 7.).

Prop.

Prop. III.

The ſquare of any ordinate to the axis, is equal to the abſciſſa multiplied into the principal latus rectum.

For (Prop. 1.) $SP=AN+AS=$ (Prop. 2.) $AN+\frac{1}{4}BC$; and $NS=AN-SA=AN-\frac{1}{4}BC$; hence (Euclid, B. 1. P. 47.), $NP^2=SP^2-SN^2=\overline{AN+\frac{1}{4}BC}^2-\overline{AN-\frac{1}{4}BC}^2=BC\times AN=$ (Prop. 2.) $4SA\times AN=L\times AN$.

Cor. Hence, BC being conſtant, AN varies as NP^2.

Prop. IV.

If PT *biſect the angle* SPM, *it will be a tangent to the parabola at* P.

FIG. 2. Join SM; then as $SP=PM$, $\angle SPn=\angle MPn$, and nP is common to the triangles SPn, MPn, $Sn=nM$, and the angles at n are right angles (Euc. B. 1. P. 4. and D. 10.). Now if PT be not a tangent, let it cut the curve in ſome other point p; join Sp, Mp, and draw mp perpendicular to mL. Then as $Mn=nS$, $\angle Mnp=\angle Snp$, and np is common to the triangles Mnp, Snp, we have $Mp=Sp$; but, by conſtruction, $mp=Sp$; therefore $Mp=mp$, or the hypothenuſe = the perpendicular, which is impoſſible; therefore PT cannot meet the curve at any other point than P, and conſequently is a tangent to it at that point.

Cor. 1. A tangent at the vertex A is perpendicular to the axis.

COR. 2. As MP is parallel to TS, the $\angle MPT = \angle PTS$, therefore $\angle PTS = \angle SPT$; hence, $SP = ST$.

FIG. 1. COR. 3. As VQ is parallel to PT, the $\angle TPX = \angle PXV$, and $\angle PVX = \angle MPT$; but $\angle MPT = \angle TPX$; therefore $\angle PVX = \angle PXV$; hence, $PX = PV$.

PROP. V.

The ſubtangent is equal to twice the abſciſſa.

For (Cor. 2. Prop. 4.) $TS = SP =$ (Prop. 1.) $AN + AS$; take away AS from both, and $TA = AN$; therefore $TN = 2AN$.

FIG. 3. COR. If VQ be produced to meet TZ in O, then the triangles TPN, OVG are ſimilar, and $NP = GV$; therefore $OG = TN = 2AN$.

PROP. VI.

The ſub-normal is equal to half the principal latus rectum.

FIG. 1. For the triangles TNP, NGP are ſimilar (EUC. B. VI. P. 8.), and hence, $TN : PN :: PN : NG = \frac{PN^2}{TN}$ = (Prop. 3. and 5.) $\frac{BC \times AN}{2AN} = \frac{1}{2}BC =$ (Prop. 2.) $\frac{1}{2}$ the principal latus rectum.

COR. As (Prop. 4. Cor. 2.) $ST = SP$, and the angle TPG is a right angle, a circle whoſe center is S, and radius ST, will paſs through P and G; hence, the angle $TSP = 2\angle TGP$ (EUC. B. III. P. 20.).

PROP. VII.

The ſquare of the ordinate to any diameter, is equal to it's parameter multiplied into the abſciſſa.

Through $\mathcal{Q}$ draw $E\mathcal{Q}D$ perpendicular to AZ, meeting WP (produced if neceſſary) in D. FIG. 3.

Now $OE^2 = \overline{OG-GE}^2 = \overline{2AN-GE}^2$ (Pr. 5. Cor.)
$AE = AN + NG - GE$.
$TN^2 = OG^2 = 4AN^2$ (Prop. 5. Cor.)
$GV^2 = NP^2 = 4AS \times AN$ (Prop. 3.)
$E\mathcal{Q}^2 = 4AS \times AE$ (Pr. 3.) $= 4AS \times \overline{AN+NG-GE}$.

By ſimilar triangles TNP, $OE\mathcal{Q}$,

$TN^2 : NP^2 :: OE^2 : E\mathcal{Q}^2$, that is,
$4AN^2 : 4AS \times AN :: \overline{2AN-GE}^2 : 4AS \times \overline{AN+NG-GE}$,
or, $4AN : 1 :: \overline{2AN-GE}^2 : AN+NG-GE$;

Multiply extremes and means, and we get

$$4AN \times NG = GE^2 = VD^2.$$

By ſimilar triangles OGV, $VD\mathcal{Q}$,

$OG^2 : GV^2 :: VD^2 : D\mathcal{Q}^2$; that is,

$4AN^2 : 4AS \times AN :: 4AN \times NG : D\mathcal{Q}^2 = 4AS \times NG$;

hence, $\mathcal{Q}V^2 = VD^2 + D\mathcal{Q}^2$ (Euc. B. 1. P. 47.) $= 4AN \times NG + 4AS \times NG = \overline{4AN+4AS} \times NG = 4SP \times PV$ (Prop. 1.).

If QV be produced to meet the curve at Q', and $Q'E'$ be perpendicular to AE', then $OE'=OG+GE'$, and $AE'=AN+NG+GE'$; and by the very ſame ſteps, we get $Q'V^2=4SP\times PV$.

COR. 1. Hence, $QV=Q'V$; therefore every line QQ' is biſected by the abſciſſa PV.

COR. 2. When the point P is given, $4SP$ is conſtant; hence, for the ſame diameter, the abſciſſa varies as the ſquare of the ordinate.

COR. 3. If QI be drawn parallel to PV, then $4SP\times QI=PI^2$; hence, when the point P is given, QI varies as PI^2.

PROP. VIII.

If a perpendicular from the focus be let fall on a tangent at any point of a parabola, it will be a mean proportional between the diſtance of that point from the focus, and the diſtance of the vertex from the focus.

FIG. 1. As (Cor. 2. Prop. 4.) $SP=ST$, the perpendicular SY muſt biſect TP, or $TY=YP$; but (Prop. 5) $TA=AN$; therefore (EUC. B. VI. P. 2.) AY is parallel to PN, and conſequently perpendicular to the axis. Since therefore SY is perpendicular to TY, and YA perpendicular to TS, the triangle SYA is ſimilar to SYT (EUC. B. VI. P. 8.), or to SYP, and $SP : SY :: SY : SA$.

COR. 1. Hence, $SP : SA :: SP^2 : SY^2$.

COR. 2. As $SP=\dfrac{SY^2}{SA}$, and SA is conſtant, SP varies as SY^2.

Cor. 3. As $SY^2 = SA \times PS$, therefore $4SY^2 = 4SA \times PS =$ (Prop. 2.) $BC \times PS$.

DEFINITION.

Let PW be any curve, Pvq a circle touching it at P, and TP a tangent at P; draw Rq cutting PW in Q; then if RQq move up to P, and the limiting ratio of $RQ : Rq$ be a ratio of equality, the circle is ſaid to be a *Circle of Curvature* to the curve PW at P. FIG. 4.

LEMMA.

Draw Pv *parallel to* Rq, *and join* vq, Pq; *then when* q *and* R *move up to* P, Pv *is equal to the ultimate value of* $\frac{PQ^2}{RQ}$.

The triangles PRq, Pvq are ſimilar, for the angle $RqP =$ the alternate angle qPv, and the angle RPq between the chord and tangent $= \angle Pvq$ in the alternate ſegment; hence, $qR : qP :: qP : Pv = \frac{qP^2}{qR}$; but by Sir I. Newton's *Principia*, Lib. 1. Lem. 7. the limit of the chord qP to the arc qP is a ratio of equality; and the limit of the arc qP to the arc PQ is a ratio of equality, becauſe (by the ſame Lemma) the limit of each to PR is a ratio of equality; alſo (by the above Definition), the limit of $qR : RQ$ is a ratio of equality; hence, when q and R move up to P, Pv is equal to the ultimate value of $\frac{PQ^2}{RQ}$.

PROP. IX.

FIG. I. *At any point* P *of a parabola, the chord* PSv *of the circle of curvature, paſſing through the focus, is equal to the latus rectum belonging to that point.*

Draw QR parallel to PS. By Prop. 7. $4SP = \frac{QV^2}{PV}$ $=$ (as $QR = PX = PV$ Prop. 4. Cor. 3.) $\frac{QV^2}{QR}$; but (NEWT. *Prin.* Lib. 1. Lem. 7. Cor.) the limiting ratio of QV to QP is a ratio of equality; therefore ultimately when Q moves up to P, $4SP = \frac{QP^2}{QR} = Pv$ by the Lemma.

COR. Draw vF perpendicular to vP, meeting PF perpendicular to PT in F, then PF is the diameter of the circle of curvature; becauſe Pv being the chord of that circle, and the angle PvF a right angle, it muſt be an angle in a ſemicircle. Draw SY perpendicular to PT; then the triangles SPY, PvF are ſimilar, being right angled at Y and v, and having the alternate angles YSP, vPF equal; hence, $SY : SP :: Pv\ (4SP) : PF = \frac{4SP^2}{SY} =$ (as $SY = \sqrt{SP \times SA}$ Prop. 8.) $\frac{4SP^{\frac{3}{2}}}{\sqrt{SA}}$.

PROP.

PROP. X.

FIG. 5. *Let two parabolas* AY, AZ, *be deſcribed on the ſame axis* AX, *to which* NPQ *is drawn perpendicular; then if the lines* PS, QS, *be drawn to any point* S *in the axis, the area* AQS *will be to the area* APS *in a given ratio.*

For (Cor. Prop. 3.) QN^2 varies as AN, and PN^2 varies as AN; hence, QN^2 varies as PN^2, conſequently QN varies as PN. Draw qpn indefinitely near to QPN, and the areas $NQqn$, $NPpn$ will be the increments of the areas AQN, APN; but $NQqn$ is to $NPpn$ ultimately as QN to PN, becauſe their baſes are equal; hence, the areas AQN, APN beginning together, and always increaſing in the given ratio of QN to PN, the areas themſelves will be in that ratio. But the triangle SQN is to the triangle SPN as QN to PN; hence, $AQN-SQN$ is to $APN-SPN$, that is, AQS is to APS, as QN is to PN, or in a given ratio. If N be between A and S, then $AQS=AQN+SQN$, and $APS=APN+SPN$, and the ſame concluſion follows.

PROP. XI.

FIG. 25. *Let the line* QV *meet the parabola* LM *in* Q, V; *draw the tangent* QS, *and let* SV *be parallel to the axis, and from any point* T *draw* TPR *parallel to* SV; *then* SV : TR :: TR : TP.

For (Prop. 7. Cor. 3.) $TP : SV :: QT^2 : QS^2 ::$ (by ſim. tri.) $TR^2 : SV^2$, therefore $TR^2 = TP \times SV$, and $SV : TR :: TR : TP$.

Cor. 1. Becauſe $TR : TP :: SV : TR ::$ (by ſim. tri.) $QV : QR$, we have, by diviſion, $PR : TP :: RV : RQ$; hence, QV, TR, are divided at R and P in the ſame ratio.

Cor. 2. If R biſect QV, then (Prop. 7. Cor. 1.) PR is an abſciſſa to the ordinate QR, and QR is parallel to a tangent PW at P (Def. 6.). Now as $RV = RQ$, $PR = TP$ (Cor. 1.); hence, $TW = WQ$.

Reaſoning in the ſame manner for a tangent at V as at Q, it appears that a tangent at V muſt paſs through T.

Prop. XII.

FIG. 6. *If a cone* HXY *be cut by a plane* FZW *parallel to a plane touching the ſide* XH *of the cone, the ſection* FZW *will be a parabola.*

Let $AECV$ be a ſection of the cone parallel to the baſe, whoſe diameter (the ſection being a circle) interſects the axis FR of the figure FZW in B, and whoſe circumference cuts the curve FZ terminating the ſection FZW, in E, and join BE; then, by the property of the circle, $AB \times BC = BE^2$. Draw GF parallel to ABC, and then $AB = GF$ a conſtant quantity; and as HX is that ſide of the cone to which FR is parallel, the triangles HGF, FBC are ſimilar, and $HG : GF :: FB : BC = \dfrac{GF \times FB}{HG}$, therefore

$GF^2 \times FB$

$\frac{GF^2 \times FB}{HG} = GF \times BC = AB \times BC = BE^2$, or $\frac{GF^2}{HG} \times FB = BE^2$; but GF and HG being conſtant quantities, the abſciſſa FB varies as the ſquare of the ordinate BE; therefore the curve is a parabola, as appears by Cor. Prop. 3.

Hence, according to our definition, the *Parabola* is a conic ſection.

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ON

ON

THE ELLIPSE.

DEFINITIONS.

FIG. 7. 1. IF two right lines *Sa*, *Hb*, revolve about two fixed points *S*, *H*, and continually interſect each other in a point *P*, ſo that $SP+HP$ may be a conſtant quantity, the curve deſcribed by the point *P* is called an *Ellipſe*.

2. The points *S* and *H* are called *Foci*.

3. The right line *AM* paſſing through the foci and terminated by the curve at *A* and *M*, is called the *Axis major*.

4. If *AM* be biſected in *C* (which is called the *Center*), and *BCD* be drawn perpendicular to the axis major, and terminated by the curve, it is called the *Axis minor*.

5. A right line *LST* paſſing through the focus *S* perpendicular to the major axis and terminated by the curve, is called the *Latus rectum*.

FIG. 8. 6. If *PCG* be a right line paſſing through the center *C*, it is called a *Diameter*. And if *DCF* be drawn parallel to a tangent RP at P, it is called a *conjugate Diameter* to the diameter PCG. Alſo, PCG, DCF, are called *conjugate Diameters.*

7. If QV be drawn parallel to a tangent at P, it is called an *Ordinate*, and GV, PV are called *Abſciſſas.*

FIG. 7. 8. If TV be a tangent at T, meeting the axis MA produced in V, and the line xVy be drawn perpendicular to MV, that line is called the *Directrix.*

FIG. 11. 9. If PG be drawn perpendicular to the ellipſe APM at any point P, and PN perpendicular to the axis AM, then PG is called the *Normal*, and NG the *Sub-normal.*

PROP. I.

The ſum of two right lines drawn from the foci to any point of the curve, is equal to the major axis.

FIG. 7. For conceiving the deſcribing point P to come to A and M, we have, in the former caſe, the ſum of the lines drawn from S and H equal to $HA+AS=SH+2AS$, and in the latter caſe, the ſum is equal to $SH+2HM$; hence (Def. 1.), $SH+2AS=SH+2HM$, conſequently $AS=HM$; hence, $HP+PS=HA+AS=HA+HM=AM$, or $=2AC$.

COR. 1. Hence, $SC=CH$.

COR. 2. By tranſpoſition, $HP=2AC-SP$.

PROP.

Prop. II.

A line drawn from either of the foci to the extremity of the axis minor, is equal to the ſemi-axis major.

For the triangles HCB, SCB, have $SC=CH$ (Cor. 1. Prop. 1.), BC common, and the angle $SCB=$ the angle HCB, therefore (Euc. B. 1. P. 4.) $SB=BH$; but (Prop. 1.) $SB+BH=2AC$; therefore $2AC=2SB$; conſequently $AC=SB=BH$.

Cor. Hence, $MS\times SA=BC^2$. For $BC^2=BS^2-SC^2=AC^2-SC^2=\overline{AC+SC}\times\overline{AC-SC}=\overline{MC+SC}\times\overline{AC-SC}=MS\times SA$.

Prop. III.

The latus rectum is a third proportional to the axis major and the axis minor.

By Cor. 2. Prop. 1. $LH=2AC-SL$; hence, $HS^2+SL^2=LH^2=4AC^2-4AC\times SL+SL^2$, therefore $HS^2+4AC\times SL=4AC^2=$ (Prop. 2.) $4SB^2$; hence, $4AC\times SL=4SB^2-HS^2=$ (becauſe $2SC=HS$) $4SB^2-4SC^2=4BC^2$, therefore $2AC:2BC::2BC:2SL$, or $AM:BD::BD:LT$.

Prop. IV.

FIG. 9. *If* SP *be produced to* m, *and the right line* VPW *biſect the angle* HPm, *it will be a tangent to the ellipſe at the point* P.

For take $Pm=PH$, and join Hm; then as $mP=PH$, $\angle mPn=\angle HPn$, and Pn common to the triangles mPn, HPn, we have $mn=Hn$, and the angles at n right angles (Euc. B. 1. P. 4. and D. 10.). Now if PW be not a tangent, let it cut the curve at ſome point $\mathcal{Q}$, and join $\mathcal{Q}H$, $\mathcal{Q}m$, $\mathcal{Q}S$. Then as $Mn=nH$, the angles at n right angles, and $n\mathcal{Q}$ common to the triangles $mn\mathcal{Q}$, $Hn\mathcal{Q}$, we have $m\mathcal{Q}=H\mathcal{Q}$. Now (Euc. B. 1. P. 20.) $S\mathcal{Q}+\mathcal{Q}m$ is greater than Sm, or greater than $SP+Pm$, or $SP+PH$, or AM (Prop. 1.); therefore $S\mathcal{Q}+\mathcal{Q}H$ is greater than AM, conſequently $\mathcal{Q}$ is not a point in the curve (Prop. 1.); therefore as VPW cannot meet the curve at any other point, it muſt be a tangent.

Cor. 1. A tangent at A, or M, is perpendicular to AM.

Cor. 2. As the angle $mPW=$ opp. angle SPV, we have the angle $HPW=$ the angle SPV; that is, two right lines drawn from the foci to any point of the curve, make equal angles with the curve.

Cor. 3. Complete the parallelogram $SPHG$; then $SG+GH=SP+PH=AM$; therefore (Def. 1.) G is a point in the ellipſe; and as the diagonals of a parallelogram biſect each other, PG paſſes through the center C of the ellipſe; hence, the center biſects all the diameters.

Cor. 4. Draw the tangent Gt at the point G. Then the $\angle SPH=\angle SGH$; but as $\angle SPV=\angle HPW$, the $\angle SPV=\frac{1}{2}$ ſupplement of $\angle SPH$; for the ſame reaſon, the $\angle tGH=\frac{1}{2}$ ſupplement of $\angle SGH$; hence, $\angle SPV=\angle tGH$; and as SP is parallel to GH, tG muſt be parallel to PV.

Cor. 5. The part PE of the line PS intercepted between any point P and the conjugate diameter DCK, is equal to AC. For draw HI parallel to FIG. 10.

DCK, and PF perpendicular to it, and conſequently, perpendicular to HI, interſecting HI in X. Then as $\angle SPV = \angle HPW$ (Prop. 4. Cor. 2.), the $\angle IPX = \angle HPX$, and PX is common to the two right angled triangles PXI, PXH; hence, $PI = PH$. But as $SC = CH$, and CE is parallel to HI, $SE = IE$, or $IE = \frac{1}{2} SI$; and as $PI = PH$, $PI = \frac{1}{2}(PI + PH)$; hence, $PE = IE + PI = \frac{1}{2}(PI + SI + PH) = \frac{1}{2}(SP + PH) = AC$ (Prop. 1.).

Prop. V.

If a circle be deſcribed on the major axis of an ellipſe, and from each of the foci perpendiculars be drawn to a tangent at any point of the ellipſe, they will meet it in the periphery of the circle.

FIG. 11. For let SY, HZ, be drawn from the two foci perpendicular to the tangent YZ at P; join YC, produce HP to W, making $PW = PS$, and join WY. Then (Prop. 4.) $\angle WPY = \angle SPY$, and as $PS = PW$, and PY is common to the triangles SPY, WPY, $WY = YS$, and $\angle PYS = \angle PYW$; but the angle PY is a right angle; therefore PYW is alſo a right angle; conſequently SYW is a ſtraight line. Now as $SY = YW$, and (Cor. 1. Prop. 1.) $SC = CH$, CY is parallel to HW; hence, $SC : CY :: SH : HW$; but $SC = \frac{1}{2} SH$; therefore $CY = \frac{1}{2} HW = \frac{1}{2}(HP + PS) =$ (Prop. 1.) $\frac{1}{2} AM = AC$; hence, as $CY = CA$, Y muſt always be in the periphery of a circle whoſe center is C and radius CA. In the ſame manner it may be proved for the other perpendicular HZ, that Z is in the periphery.

PROP. VI.

The rectangle under the perpendiculars from the foci upon the tangent, is equal to the ſquare of the ſemi-axis minor.

For produce ZH to meet the circle in E, and join CE; then (Prop. 5.) Z being in the periphery of the circle, and the angle HZY a right angle, the angle YZE muſt be in a ſemicircle; conſequently YCE muſt be a diameter. Hence, in the triangles YCS, ECH, $YC=EC$, $SC=HC$, and $\angle YCS=\angle ECH$, therefore $EH=YS$; but by the property of the circle, $ZH\times HE=MH\times HA=$ (Cor. Prop. 2.) BC^2; hence, $ZH\times SY=BC^2$.

COR. 1. Becauſe (Pr. 4. Cor. 2.) $\angle SPY=\angle HPZ$, the triangles SPY, HPZ are ſimilar, and $SP:SY::HP:HZ=\dfrac{SY\times HP}{SP}$; hence, $\dfrac{SY^2\times HP}{SP}=SY\times HZ=BC^2$; therefore $SY^2=BC^2\times\dfrac{SP}{HP}$.

COR. 2. As BC is conſtant, SY varies as $\sqrt{\dfrac{SP}{HP}}$; but $HP=$ (Cor. 2. Prop. 1.) $2AC-SP$; therefore SY varies as $\sqrt{\dfrac{SP}{2AC-SP}}$, or $\dfrac{1}{SY}$ varies as $\sqrt{\dfrac{2AC-SP}{SP}}$.

Cor. 3. Becauſe $4SY^2 = \frac{4BC^2 \times SP}{2AC - SP}$, and (Prop. 3.) $4BC^2 = 2AC \times L$, therefore $4SY^2 = \frac{2AC \times L \times SP}{2AC - SP}$; hence, $L \times SP : 4SY^2 :: L \times SP : \frac{2AC \times L \times SP}{2AC - SP} ::$ $2AC - SP : 2AC$, and as $2AC - SP$ is leſs than $2AC$, $L \times SP$ is leſs than $4SY^2$.

Prop. VII.

If an ordinate be drawn to the major axis, the product of the abſciſſas is to the ſquare of the ordinate, as the ſquare of the ſemi-axis major to the ſquare of the ſemi-axis minor.

FIG. 12. Draw PN perpendicular to AM, then (Def. 7. and Cor. 1. Prop. 4.) PN is an ordinate to the axis AM; take $AD = PS$; then

$$PS = AC - CD$$
$$\therefore PH = AC + CD \text{ (Pr. 1.)}$$

Alſo, $NS = CS - CN$

$$NH = CH + CN = CS + CN \text{ (Pr. 1. Cor. 1.)}$$

but $PH^2 = PN^2 + NH^2$ (Euc. B. 1. P. 47.)

$PS^2 = PN^2 + NS^2$; that is,

$$\overline{AC + CD}^2 = PN^2 + \overline{CS + CN}^2$$
$$\overline{AC - CD}^2 = PN^2 + \overline{CS - CN}^2$$

ſub. and $AC \times CD = CS \times CN$,

therefore $CD = \frac{CS \times CN}{AC}$; hence,

$$PH = AC + CD = AC + \frac{CS \times CN}{AC} = \frac{AC^2 + CS \times CN}{AC};$$

$$PS = AC - CD = AC - \frac{CS \times CN}{AC} = \frac{AC^2 - CS \times CN}{AC}.$$

But $PS^2 = PN^2 + NS^2$ (Euc. B. 1. P. 47.); that is, $\frac{AC^4 - 2AC^2 \times CS \times CN + CS^2 \times CN^2}{AC^2} = PN^2 + CS^2 - 2CS \times CN + CN^2$, multiply both ſides of the equation by AC^2, and we get, by tranſpoſition, $AC^2 \times PN^2 = AC^4 - AC^2 \times CN^2 - CS^2 \times AC^2 + CS^2 \times CN^2 = \overline{AC^2 - CN^2} \times \overline{AC^2 - CS^2}$ = (as $AC = SB$) $\overline{AC^2 - CN^2} \times \overline{SB^2 - CS^2}$ (BC^2) = $\overline{AC + CN} \times \overline{AC - CN} \times BC^2 = \overline{MC + CN} \times \overline{AC - CN} \times BC^2 = MN \times AN \times BC^2$, conſequently $MN \times AN : PN^2 :: AC^2 : BC^2$.

Cor. 1. Draw Pn perpendicular to BCE. Now by the propoſition,

$AN \times NM : NP^2 :: AC^2 : CB^2$

or $\overline{AC - nP} \times \overline{AC + nP}$ $(AC^2 - nP^2) : Cn^2 :: AC^2 : CB^2$

alt. $AC^2 - nP^2 : AC^2 :: Cn^2 : CB^2$

div. $nP^2 : AC^2 :: CB^2 - Cn^2 : CB^2$

alt. $nP^2 : \overline{CB + Cn} \times \overline{CB - Cn}$ $(CB^2 - Cn^2) :: AC^2 : CB^2$

inv. $En \times Bn : nP^2 :: CB^2 : AC^2$

Hence, there is the ſame relation between the abſciſſas and ordinates of the minor axis, as between thoſe of the major axis.

Cor. 2. If a circle $ARMK$ be deſcribed on the major axis AM of an ellipſe, as a diameter, and NP be produced to Q, NP ſhall be in a given ratio to NQ. FIG. 11.

NQ. For by the property of the circle $AN \times MN = NQ^2$; hence,

$$NQ^2 : NP^2 :: AC^2 : BC^2,$$

and $NQ : NP :: AC : BC :: AM : BD$ a given ratio.

And by Cor. 1. it appears, that the ſame muſt be true, if AM be the minor axis of the ellipſe.

Cor. 3. Becauſe $PN : QN$ in the conſtant ratio of $BD : AM$ (Cor. 2.), therefore for the ſame reaſon as in the *Parabola*, Prop. 10. the area APN : the area $AQN :: PN : QN :: BD : AM$. Hence, the whole area of the ellipſe : the whole area of the circle $:: BD : AM$.

Cor. 4. Join QH, PH, then the triangles HNP, HNQ are to each other as $NP : NQ$, or as $BD : AM$; whence, and the laſt Cor. we have the area APH : the area $AQH :: BD : AM$. And the ſame muſt evidently be true, if H be aſſumed any where in the axis major.

Cor. 5. Conceive a circle to be deſcribed about C with a radius $= \sqrt{AC \times BC}$, a mean proportional between the ſemi-axes of the ellipſe, and let a be it's area; and let $A =$ the area of the circle $ARMK$; then (Cor. 4.)

A : area $ABMD$ of ellipſe $:: AC : BC$

$a : A :: AC \times BC : AC^2$ (Euc. B. xii. P. 2.)

$\therefore a$: area $ABMD$ of ellipſe $:: 1 : 1$;

hence, $a =$ the area $ABMD$ of the ellipſe.

Cor. 6. As (Cor. 3.) the whole area (A) of the ellipſe is to the whole area (B) of the circle, as the axis minor (a) is to the axis major (b), we have $A = \frac{B \times a}{b}$; but B varies as b^2; therefore A varies as $b \times a$; that is, the area of an ellipſe varies as the product of the axis major and minor; but the area of a parallelogram circumſcribing the ellipſe about the axis major and minor, varies in the ſame ratio; hence, the area of every ellipſe is in a given ratio to the area of the parallelogram deſcribed about the major and minor axes.

Cor. 7. If $L =$ the latus rectum, then (Prop. 3.) $AM : BD :: BD : L$; hence, $AM : L :: AM^2 : BD^2 :: AC^2 : BC^2 ::$ (by this prop.) $MN \times AN : PN^2$, conſequently $L = \frac{AM \times PN^2}{MN \times AN} = \frac{2AC \times PN^2}{MN \times AN}$.

Cor. 8. Hence, $\frac{2AC}{L} \times PN^2 = MN \times AN = AC^2 - CN^2$ by the demonſtration of the propoſition.

Cor. 9. If the major axis of an ellipſe be increaſed without limit, the ellipſe, at all finite diſtances, approaches to a parabola as it's limit. For if AN remain finite, and AM, NM be increaſed without limit, their difference AN becomes indefinitely ſmall in reſpect to the quantities themſelves; therefore the *limit* of AM, NM is a ratio of equality; but $L = \frac{AM \times PN^2}{MN \times AN}$; therefore the limit of this is $L = \frac{PN^2}{AN}$, or $L \times AN = PN^2$, which is the property of the parabola, by Prop. 3. of the Parabola.

PROP. VIII.

FIG. 14. *If a circle be deſcribed on the major or minor axis of an ellipſe, and (when neceſſary)* NP *be produced to meet it at* Q, *the tangents to the circle and ellipſe at the points* Q *and* P *will interſect the axis* MA *produced, in the ſame point* T.

For draw QT a tangent to the circle at Q, and join TP, and if it be not a tangent to the ellipſe, let it cut it in ſome other point K, and draw the ordinate LK, which (when neceſſary) produce to meet the circle at I, and it's tangent at O, then the triangles TNP, TLK, and TNQ, TLO, are ſimilar; hence,

$TN : TL :: NP : LK$,
and $TN : TL :: NQ : LO$,
$\therefore\ NP : LK :: NQ : LO$,
alt. $NP : NQ :: LK : LO$,
but $NP : NQ :: LK : LI$ (Cor. 2. Pr. 7.)

therefore $LO = LI$, which is abſurd; hence, the line TP does not cut the ellipſe, and conſequently is a tangent to it.

COR. By EUC. B. VI. P. 8. the triangles CNQ, CQT are ſimilar, therefore $CN : CQ\ (CA) :: CQ\ (CA) : CT$; alſo, as $CA = CM$, $CN : CM :: CM : CT$.

PROP.

PROP. IX.

The ſub-normal NG $= \frac{L}{2AC}\times CN$. FIG. 11.

For (Prop. 5.) SW is perpendicular to PY, and PG is alſo perpendicular to PY, it being (Def. 9.) perpendicular to the curve at P; therefore the triangles HWS, HPG are ſimilar, and $HW : HS :: HP : HG = \frac{HS\times HP}{HW}$; but (Prop. 7.) $HP = AC + \frac{CS\times CN}{AC}$; alſo (Prop. 1. Cor. 1.), $HS = 2CS$; and (Prop. 5.) $HW = 2AC$; therefore $HG = \frac{2CS}{2AC}\times\overline{AC + \frac{CS\times CN}{AC}} = \frac{CS\times AC^2 + CS^2\times CN}{AC^2}$; hence, $CG = HG - HC = HG - CS = \frac{CS\times AC^2 + CS^2\times CN}{AC^2} - CS = \frac{CS^2\times CN}{AC^2}$; conſequently $NG = CN - CG = CN - \frac{CS^2\times CN}{AC^2} = \frac{AC^2 - CS^2}{AC^2}\times CN =$ (as $AC = BS$ Prop. 2.) $\frac{BS^2 - CS^2}{AC^2}\times CN = \frac{BC^2}{AC^2}\times CN =$ (as $\frac{BC^2}{AC^2} = \frac{L}{2AC}$ Prop. 7. Cor. 7.) $\frac{L}{2AC}\times CN$.

COR. By ſimilar triangles GPN, PNT, $\frac{L}{2AC}\times CN$ FIG. 14.

$(NG) : PN :: PN : NT = \frac{2AC \times PN^2}{L \times CN} =$ (Prop. 7. Cor. 8.) $\frac{AC^2 - CN^2}{CN}$.

PROP. X.

FIG. 14. *If* CH *be a ſemi-conjugate to* CP, *and* HR *be an ordinate to the axis* AM, *then* $CN^2 = AC^2 - CR^2$.

As CH is parallel to PT (Def. 6.), the triangles CHR, TPN are ſimilar, and $TN^2 : NP^2 :: CR^2 : RH^2$, or (Prop. 9. Cor. and Prop. 7. Cor. 8.) $\frac{\overline{AC^2 - CN^2}^2}{CN^2} : \overline{AC^2 - CN^2} \times \frac{L}{2AC} :: CR^2 : \frac{L}{2AC} \times \overline{AC^2 - CR^2}$, or $AC^2 - CN^2 : CN^2 :: CR^2 : AC^2 - CR^2$, therefore by comp. $AC^2 : CN^2 :: AC^2 : AC^2 - CR^2$, conſequently $CN^2 = AC^2 - CR^2$.

COR. Hence, (Pr. 7. Cor. 8.) $\frac{2AC}{L} \times PN^2 = AC^2 - CN^2 = CR^2$; but (Prop. 3.) $2AC : 2BC :: 2BC : L$, therefore $2AC : L :: AC^2 : BC^2$; hence, $\frac{AC^2}{BC^2} \times PN^2 = CR^2$, and $\frac{AC}{BC} \times PN = CR$. For the ſame reaſon, $\frac{AC}{BC} \times HR = CN$.

PROP. XI.

If at the extremities of any two conjugate diameters, tangents be drawn to the ellipſe, they will form a parallelogram; and all ſuch parallelograms will be equal.

Let PG, DK be any two conjugate diameters; FIG. 18. then (Prop. 4. Cor. 4.) the tangent aPd is parallel to the tangent bGc, and the tangent aDb is parallel to the tangent dKc, therefore $abcd$ is a parallelogram. Let ACM be the axis-major, CB the ſemi-minor axis; produce AM to meet the tangent at P in T, and draw Cr perpendicular to ad. By Prop. 10. Cor. $\frac{AC}{BC} \times KR = CN$, therefore $\frac{KR}{CN} = \frac{BC}{AC}$; and by ſimilar triangles CKR, CTr, $CK : KR :: CT$ ($= \frac{CA^2}{CN}$ Prop. 8. Cor.) : Cr, therefore $CK \times Cr = CA^2 \times \frac{KR}{CN} = CA \times BC$, a conſtant quantity, and $CK \times Cr =$ the area $CKdP$ which is one fourth of $abcd$; therefore the area $abcd$ is conſtant.

COR. 1. From hence, and Cor. 6. Prop. 7. the area of the ellipſe varies as the area of the circumſcribing parallelogram $abcd$; and as the area $aPCD$ of a fourth part of the parallelogram is equal to $DC \times PF$ (PF being perpendicular to DCK), the whole parallelogram varies as $DC \times PF$; therefore the area of the ellipſe varies as $DC \times PF$.

COR. 2. Becauſe the triangles SPY, HPZ, PFE FIG. 10. are ſimilar, we have,

SP

$$SP : SY :: PE : PF$$
$$HP : HZ :: PE : PF$$
$$\therefore SP \times HP : SY \times HZ :: PE^2 : PF^2;$$

but (Prop. 6.) $SY \times HZ = BC^2$, and (Cor. 5. Pr. 4.) $PE^2 = AC^2$; therefore $SP \times HP = \frac{BC^2 \times AC^2}{PF^2} = DC^2$ by the Propoſition.

Prop. XII.

If an ordinate be drawn to any diameter, the rectangle under the abſciſſas is to the ſquare of the ordinate, as the ſquare of the ſemi-diameter to the ſquare of the ſemi-conjugate diameter.

FIG. 16. Draw the ordinate Mv parallel to PT or CK, and MQ, vn perpendicular to ACE, alſo ev parallel to nQ, and Pg perpendicular to PT. Then as Mv, ve, Me are reſpectively parallel to PT, TN, PN, the triangle Mev is ſimilar to PNT, it is therefore ſimilar to the triangles PNg, and CKR; hence, $PN : Ng$ ($= \frac{L \times CN}{2AC}$, Prop. 9.) $:: ve : eM = \frac{L \times CN \times ve}{2AC \times NP}$; alſo, by ſimilar triangles CPN, Cvn, $CN : NP :: Cn : nv = \frac{Cn \times NP}{CN}$; therefore $QM = eM + eQ = eM + nv = \frac{L \times CN \times ve}{2AC \times NP} + \frac{Cn \times NP}{CN}$. Alſo, $CQ = nQ - Cn = ve - Cn$. Now $\frac{2AC}{L} \times QM^2 = AC^2 - CQ^2$ (Prop. 7.

Cor. 8.), in which equation, ſubſtitute for $\mathcal{Q}M$ and $C\mathcal{Q}$ the values found above, and we have, $\frac{2AC}{L} \times \left(\frac{L^2 \times CN^2 \times ve^2}{4AC^2 \times NP^2} + \frac{Cn^2 \times NP^2}{CN^2} + \frac{L \times ve \times Cn}{AC}\right) = AC^2 - ve^2 + 2ve \times Cn - Cn^2$; but the laſt term on the firſt ſide, and the third term on the ſecond ſide deſtroy each other, therefore the equation becomes $\frac{L \times CN^2 \times ve^2}{2AC \times NP^2} + \frac{2AC \times Cn^2 \times NP^2}{L \times CN^2} = AC^2 - ve^2 - Cn^2$. But $\frac{2AC}{L} \times PN^2 = AC^2 - CN^2$ (Prop. 7. Cor. 8.); in the laſt equation therefore, for CN^2 in the firſt term, put $AC^2 - \frac{2AC}{L} \times PN^2$, and for NP^2 in the ſecond term, put $\frac{L}{2AC} \times \overline{AC^2 - CN^2}$, and we get $\frac{L \times AC \times ve^2}{2NP^2} - ve^2 + \frac{AC^2 \times Cn^2}{CN^2} - Cn^2 = AC^2 - ve^2 - Cn^2$, or $\frac{L}{2AC} \times \frac{ve^2}{PN^2} + \frac{Cn^2}{CN^2} = 1$; but $\frac{L}{2AC \times PN^2} = \frac{1}{CR^2}$ (Prop. 10. Cor.); hence, $\frac{ve^2}{CR^2} + \frac{Cn^2}{CN^2} = 1$, and $CR^2 \times \overline{CN^2 - Cn^2} = ve^2 \times CN^2$, therefore $CN^2 - Cn^2 : CN^2 :: ve^2 : CR^2 ::$ (by ſim. tri.) $vM^2 : CK^2$; alſo, $CN^2 : Cn^2 :: CP^2 : Cv^2$, and $CN^2 - Cn^2 : CN^2 :: CP^2 - Cv^2 : CP^2$; hence, from this and the firſt proportion, $CP^2 - Cv^2 : CP^2 :: vM^2 : CK^2$; but $CP^2 - Cv^2 = \overline{CP + Cv} \times \overline{CP - Cv} = Gv \times Pv$; therefore, by alternation, $Gv \times Pv : vM^2 :: CP^2 : CK^2$.

COR. Produce Mv to m; then by the ſame reaſoning it appears that $Gv \times Pv : vm^2 :: CP^2 : CK^2$; conſequently $vM = vm$, or any line Mm parallel to a tangent at P, is biſected by PG.

PROP. XIII.

The ſum of the ſquares of any two ſemi-conjugate diameters, is equal to the ſum of the ſquares of the two ſemi-axes.

FIG. 10. Draw PN perpendicular to AM; then (EUC. B. II. P. 12. 13.)

$$PS^2 = PC^2 + CS^2 + 2CS \times CN$$

$$\underline{PH^2 = PC^2 + CS^2\ (CH^2) - 2CS\ (CH) \times CN}$$

$$\therefore\ PS^2 + PH^2 = 2PC^2 + 2CS^2$$

$$\text{and } 2PC^2 = PS^2 + PH^2 - 2CS^2$$

$$\text{but } \underline{2CD^2 = 2SP \times PH \text{ (Prop. 11. Cor. 2.)}}$$

$$\therefore\ 2PC^2 + 2CD^2 = \overline{PH + PS}^2 - 2CS^2 = \text{(Prop. 1.)}$$

$4AC^2 - 2CS^2 =$ (Prop. 2.) $4SB^2 - 2CS^2 = 4SB^2 - 2SB^2 + 2CB^2 = 2SB^2 + 2CB^2 = 2AC^2 + 2BC^2$; hence, $PC^2 + CD^2 = AC^2 + BC^2$.

PROP. XIV.

FIG. 8. *If* QV *be an ordinate to* PG, *and a tangent* QT *at the point* Q, *meet the diameter* GP *produced in* T; *then* CV : CP :: CP : CT.

Draw Vr perpendicular to GP and equal to VQ, then the curve PrG paſſing through all the points r, is an ellipſe; for $GV \times VP$ being to QV^2 in a given ratio (Prop. 12.), we have $GV \times VP$ to Vr^2 in a given ratio, therefore (Prop. 7. and Cor. 1.) PrG is an ellipſe, and PG is it's major or minor axis, the ordinates being perpendicular to it. Draw rT a tangent at r; then (Prop. 8. Cor.) $CV : CP :: CP : CT$; join TQ, and if it be not a tangent at Q, let it cut the ellipſe at ſome point m; draw the ordinates ma, av, and produce av to meet Tr produced in T. Now by ſimilar triangles,

$$TV : Ta :: VQ : am$$
$$TV : Ta :: Vr\ (VQ) : at.$$

Therefore $am = at$; but by conſtruction, $am = av$; therefore $at = av$, which is abſurd; hence, TQ does not cut the curve at Q, and conſequently is a tangent to it.

Prop. XV.

If PF, *perpendicular to* DCK, *cut* AM *in* v, *then* $PF \times Pv = BC^2$.

FIG. 18.

For produce CB to meet the tangent at P in m, and draw Cr perpendicular to the tangent, and Pn perpendicular to BC; then as Cm is parallel to PN, and Cr to Pv, the triangles mCr, PvN are ſimilar, and $Cm : Cr\ (PF) :: Pv : PN\ (Cn)$; therefore $PF \times Pv = Cm \times Cn = CB^2$ (Prop. 8. Cor.).

PROP. XVI.

FIG. 7.

The diſtance $\text{SP} = \dfrac{\text{BC}^2}{\text{MC} - \text{SC} \times \textit{coſ.}\ \text{PSM}}$.

Draw PN perpendicular to AM. By Prop. 7. $MC^2 : CB^2 :: AN \times NM : PN^2$; but $CB^2 = SB^2 - SC^2 = MC^2 - SC^2$, and $AN \times NM = \overline{MC + CN} \times \overline{MC - CN} = MC^2 - CN^2 = MC^2 - \overline{SN \mp SC}^2$; hence, $MC^2 : \overline{MC^2 - SC^2} :: \overline{MC^2 - \overline{SN \mp SC}^2} : PN^2 =$

$$\frac{\overline{MC^2 - SC^2} \times \overline{MC^2 - \overline{SN \mp SC}^2}}{MC^2} =$$

$$\frac{MC^4 - MC^2 \times SN^2 \pm 2SN \times SC \times MC^2 - 2MC^2 \times SC^2 + SC^2 \times SN^2 \mp 2SN \times SC^3 + SC}{MC^2}$$

add SN^2 to both ſides of the equation, and we have $SP^2\ (PN^2 + SN^2) =$

$$\frac{MC^4 \pm 2SN \times SC \times MC^2 - 2MC^2 \times SC^2 + SC^2 \times SN^2 \mp 2SN \times SC^3 + SC^4}{MC^2},$$

the ſquare root of which is $SP = \dfrac{MC^2 \pm SN \times SC - SC^2}{MC}$ $=$ (as $MC^2 - SC^2 = SB^2 - SC^2 = BC^2$) $\dfrac{BC^2 \pm SN \times SC}{MC}$; but $\pm SN = SP \times \text{coſ.}\ PSM$; therefore $SP = \dfrac{BC^2 + SC \times SP \times \text{coſ.}\ PSM}{MC}$; hence, $SP = \dfrac{BC^2}{MC - SC \times \text{coſ.}\ PSM}$.

PROP. XVII.

FIG. 12.

If a line be drawn from the focus to any point of the curve, and from that point a line be drawn perpendicular to the directrix, theſe lines are in a conſtant ratio.

Draw PL perpendicular to xy. Now $CV = \frac{CA^2}{SC}$ (Pr. 8. Cor. and Def. 8.), $\therefore SV = \frac{CA^2}{SC} - SC = \frac{CA^2 - SC^2}{SC} = \frac{SB^2 - SC^2}{SC} = \frac{CB^2}{SC}$; and by Trig. rad. $= 1$: coſ. PSM :: SP, or $\frac{BC^2}{MC - SC \times \text{coſ. } PSM}$ (Prop. 16.), : $SN = \frac{BC^2 \times \text{coſ. } PSM}{MC - SC \times \text{coſ. } PSM}$; therefore $PL = VN = SV + SN = \frac{CB^2}{SC} + \frac{BC^2 \times \text{coſ. } PSM}{MC - SC \times \text{coſ. } PSM} = \frac{MC \times CB^2}{SC \times \overline{MC - SC \times \text{coſ. } PSM}}$; hence, $SP : PL :: \frac{BC^2}{MC - SC \times \text{coſ. } PSM} : \frac{MC \times CB^2}{SC \times \overline{MC - SC \times \text{coſ. } PSM}}$ :: $SC : MC$ a conſtant ratio.

COR. Hence, $SP : PL :: SA : AV :: SM : MV$; therefore PL is greater than PS.

PROP.

PROP. XVIII.

FIG. 7. *If the ellipſe* ABMD *be very nearly a circle, as* P *moves from* B *to* M, *the increaſe of* CP *is very nearly in proportion to the ſquare of the coſine of the angle* PCM.

By Prop. 7. $CM^2 : CB^2 :: MC^2 - CN^2 : NP^2 = \frac{CB^2}{CM^2} \times \overline{MC^2 - CN^2}$, therefore CP^2 $(= CN^2 + NP^2)$ $= CN^2 + \frac{CB^2}{CM^2} \times \overline{MC^2 - CN^2} = CB^2 - \frac{CB^2 \times CN^2}{CM^2} + CN^2 = CB^2 + \overline{CM^2 - CB^2} \times \frac{CN^2}{CM^2} = CB^2 + \overline{CM + CB} \times \overline{CM - CB} \times \frac{CN^2}{CM^2} = CB^2 + 2CB \times \overline{CM - CB} \times \frac{CN^2}{CM^2}$ very nearly, and by taking the ſquare root, we get $CP = CB + \overline{CM - CB} \times \frac{CN^2}{CM^2}$[*] very nearly, $= CB + \overline{CM - CB} \times \frac{CN^2}{CP^2}$ very nearly, $= CB + \overline{CM - CB} \times \overline{\text{coſ. } PCM}^2$, radius being unity; the quantity therefore $\overline{CM - CB} \times \overline{\text{coſ. } PCM}^2$, by which CP exceeds CB, varies as $\overline{\text{coſ. } PCM}^2$ very nearly.

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[*] That this is the root, appears from hence, that if x be very ſmall in reſpect to a, the ſquare root of $a^2 + 2ax$ is $a + x$ very nearly; for $\overline{a + x}^2 = a^2 + 2ax + x^2 = a^2 + 2ax$ very nearly.

Prop. XIX.

If a right line QSP *paſs through the focus* S *of an ellipſe, and be terminated by the curve, then* $\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{SL}$. FIG. 7.

Draw LF, nQG, TE, PmK, perpendicular to xy, and produce (if neceſſary) ST to meet PK in m; then (Prop. 17.)

$$SQ : SL :: QG : LF,$$
$$SP : ST\,(SL) :: PK : TE;$$

$$\therefore\ SL - SQ : SL :: LF - QG\ (Qn) : LF,$$
and $SP : SP - SL :: PK : PK - TE\ (Pm)$;
but $SL : SP :: LF : PK$ (Prop. 17.)

$\therefore$ by com. $SL - SQ : SP - SL :: Qn : Pm$
$:: SQ : SP$ (by ſim. tri.);

hence, $SL \times SP - SP \times SQ = SP \times SQ - SL \times SQ$,

therefore $SL \times \overline{SP + SQ} = 2SP \times SQ$, and $\frac{2}{SL} = \frac{SP + SQ}{SP \times SQ}$ $= \frac{1}{SP} + \frac{1}{SQ}$.

PROP. XX.

FIG. 17. *To determine the diameter of the circle of curvature at any point of an ellipſe, and the chords thereof which paſs through the center and focus of the ellipſe.*

Let $PaVL$ be the circle of curvature at the point P, Qv an ordinate to the diameter PG, and QR parallel to PG; then (Prop. 12.), $Pv\ (QR) \times Gv : Qv^2 :: PC^2 : CD^2$, therefore $\frac{Qv^2}{QR} = \frac{CD^2 \times Gv}{PC^2} =$ (if Q move to P, in the ultimate ſtate of the arc QP) $\frac{CD^2 \times 2PC}{PC^2} = \frac{2CD^2}{PC}$. But (NEWTON's *Prin.* Lib. 1. Lem. 7. Cor.) the limit of the ratio of $Qv : QP$ is a ratio of equality; therefore $\frac{2CD^2}{PC} = \frac{QP^2}{QR}$ ultimately; hence (Lem. pag. 7.), $\frac{2CD^2}{PC} = PV$ the *chord of curvature* paſſing through the *center* C. Alſo, if PFL be perpendicular to PR, or DK, it muſt paſs through the center of the circle; and by ſimilar triangles PCF, PVL, $PF : PC :: PV \left(\frac{2CD^2}{PC}\right) : PL = \frac{2CD^2}{PF} =$ the *diameter* of curvature. Laſtly, if PS interſect the circle in a, and La be joined, by ſimilar triangles PFE, PaL, PE ($= AC$ by Cor. 5. Prop. 4.) $: PF :: PL \left(= \frac{2CD^2}{PF}\right) : Pa = \frac{2CD^2}{AC} =$ the *chord of curvature* paſſing through the *focus*.

PROP. XXI.

If a cone ABC *be cut through it's two ſides by a plane* PMNEZ, *the ſection will be an ellipſe.* FIG. 19.

Biſect PE in K, and let a ſection $LMIZ$ be drawn through K parallel to the baſe, and alſo another ſection $HNFG$ parallel to $LMIZ$, then will theſe ſections be circles; hence, $HG \times GF = GN^2$, and $LK \times KI = KM^2$. Draw PQ, DE parallel to LI, or HF; then, by ſimilar triangles,

$$\begin{array}{l} PK : KL :: PG : GH \\ EK : KI :: GE : GF \\ \hline \therefore PK \times EK : KL \times KI :: PG \times GE : GH \times GF, \end{array}$$

that is, $PK^2 : KM^2 :: PG \times GE : GN^2$, which is the property of the ellipſe by Prop. 7.

Hence, according to our definition, the *Ellipſe* is a conic ſection.

* * *

Propoſitions 11. and 12. have here been demonſtrated directly from the preceding propoſitions, in a manner conformable to the demonſtrations there made uſe of; but in the firſt edition of this work, they were demonſtrated from the principles of projection; and as the demonſtrations are, upon ſuch principles, rendered very ſimple, we ſhall here inſert them.

FIG. Let $ARMS$ be a circle inclined to the plane of the
13. paper, and let it be projected into the curve $ABME$, by lines perpendicular to that plane; and let b be the point into which a is projected, and draw aH, bH perpendicular to AM, the interſection of the circle and the before-mentioned plane. Then, by Trigonometry, $Ha : Hb :: \text{rad.} : \text{coſ.} \angle aHb$, a conſtant ratio; therefore as, by projection, each line is diminiſhed in this ratio, the whole area will be diminiſhed in the ſame ratio. And the ſame is manifeſtly true, whatever be the figure of the plane to be projected.

By the property of the circle, $AH \times HM = Ha^2$; therefore $AH \times HM$ varies as Hb^2, which is the property of an ellipſe by Prop. 7.; hence, $AM\ E$ is an ellipſe.

Now all the ſquares deſcribed about the ſame circle are equal; and if the circle be projected into an ellipſe, every ſquare will be projected into a circumſcribing parallelogram; for the oppoſite ſides of the ſquare being parallel and tangents to the circle, they will be projected into parallel lines, which will be tangents to the ellipſe; and theſe lines will be equal; for as they are equal before projection, and equally inclined to the plane of projection, they muſt be equal in the projection, from what is proved above. And as every ſquare is diminiſhed in the conſtant ratio of radius : coſine of inclination, all the parallelograms muſt be equal, as is proved in Prop. 11.

Let $PVCG$ be a diameter of the circle, and let it be projected into a diameter $pvCg$ of the ellipſe, and let VQ, an ordinate of the circle, be projected into vq; then as VQ is parallel to a tangent at P, and parallel lines are projected into parallel lines, qv is parallel to a tangent to the ellipſe at p, for a tangent to the circle muſt evidently be projected into a tangent to the ellipſe; hence, qv is an ordinate to the abſciſſas pv, gv, Def. 7. And if DCK be drawn parallel to qv, it will be the conjugate diameter to the diameter pg. Now as all the diameters of the circle are biſected by the center, all the diameters of the ellipſe are biſected by the center, for the different parts of the ſame right line in the circle, are diminiſhed in the ſame ratio; and for the ſame reaſon, as PV, VG are projected into pv, vg, theſe latter quantities will be to the former in a given ratio, therefore $PV \times VG$ varies as $pv \times vg$; alſo, QV is to qv in a given ratio, therefore QV^2 varies as qv^2. But by the property of the circle, $PV \times VG = QV^2$; therefore $pv \times vg$ varies as qv^2; hence, $pv \times vg : qv^2 :: pC \times Cg$, or pC^2, $: CK^2$, as is proved in Prop. 12.

ON

THE HYPERBOLA.

DEFINITIONS.

FIG. 20. 1. IF two right lines Sa, Hb, interſecting each other in P, be made to revolve about H and S, ſo that $HP - SP$ may be a conſtant quantity, their common interſection P will deſcribe a curve called the *Hyperbola*. And if $Sp - Hp = HP - SP$, the hyperbola generated by the two lines Sp, Hp revolving about S and H, is called an *oppoſite* hyperbola.

2. The points H, S, are called *Foci*.

3. If A and M be the points where the curves interſect the right line HS, the line AM is called the *Axis major*.

4. Let AM be biſected in C (which is called the *Center*), and BCN be drawn perpendicular to AM; then if from A as a center, with a radius SC, a circle be deſcribed cutting BN in B and N, BN is called the *Axis minor*. And when $AM = BN$, the hyperbola is called *Equilateral*.

5. If, in like manner, two hyperbolas *dBc*, *qNs*, be deſcribed about the two axes *BN*, *AM*, they are called *conjugate* hyperbolas.

6. A right line *LST* drawn perpendicular to *HS* and terminated by the curve, is called the *Latus Rectum*.

7. If *PCG* be a right line paſſing through the center, it is called a *Diameter*.

8. If *QV* be drawn parallel to a tangent at *P*, meeting *CP* produced in *V*, then *QV* is called an *Ordinate*, and *GV*, *PV*, are called *Abſciſſas*.

9. An *Aſymptote* is a right line, which continually approaching the curve, arrives nearer to it than by any aſſignable quantity, but if indefinitely produced, never meets it.

COR. Hence, we may conſider the aſymptote as the limit to which the tangent approaches, when the abſciſſa is increaſed without limit.

10. If *TV* be a tangent at *T*, meeting the axis *AM* in *V*, and the line *xVy* be drawn perpendicular to *AM*, that line is called the *Directrix*. FIG. 15.

PROP. I.

The difference of two lines drawn from the foci to any point of the curve, is equal to the major axis.

For conceiving the deſcribing points *P*, *p*, to come to *A* and *M*, we have $HA-AS=HP-PS=Sp-pH=SM-MH$, take from the firſt and laſt of theſe, the common part *MA*, and we have $MH-AS=AS-MH$, therefore $2MH=2AS$, and $MH=AS$; hence, FIG. 20.

hence, $HP - PS = HA - AS = HA - MH = MA$, or $2AC$.

Cor. 1. Hence, $SC = CH$.

Cor. 2. By tranſpoſition, $HP = 2AC + SP$.

Prop. II.

The rectangle under the diſtances of either of the foci from the two vertices, is equal to the ſquare of the ſemi-axis minor.

For $BC^2 = BA^2 - AC^2 =$ (by conſtruction) $SC^2 - AC^2 = \overline{SC + AC} \times \overline{SC - AC} = \overline{SC + CM} \times SA = SM \times SA$.

Prop. III.

The latus rectum is a third proportional to the axis major and the axis minor.

By Cor. 2. Prop. 1. $HL = 2AC + LS$; hence, $HS^2 + SL^2$ $(= HL^2)$ $= 4AC^2 + 4AC \times LS + LS^2$, therefore $HS^2 - 4AC^2 = 4AC \times LS$; but $HS = 2CS =$ (by conſtruction) $2AB$; hence, $4AB^2 - 4AC^2 = 4AC \times LS$, or $4BC^2 = 4AC \times LS$; therefore $2AC : 2BC :: 2BC : 2LS$, that is, $AM : BN :: BN : LT$.

Prop. IV.

FIG. 21. *If* PV *biſect the angle* SPH, *it will be a tangent to the curve at* P.

For take $Pm = PS$, and join Sm; then as $mP = PS$, $\angle mPn = \angle SPn$, and Pn is common to the triangles mPn, SPn, we have $mn = Sn$, and the angles at n right angles (Euc. B. 1. P. 4). Now if PV be not a tangent, let it cut the curve at ſome point Q, and join QS, Qm. Then as $mn = nS$, and the angles at n are right angles, and nQ common to the triangles mnQ, SnQ, we have $mQ = SQ$. Now (Prop. 1) $MA = HP - PS = HP - Pm = Hm$; alſo, $Hm + mQ$ is greater than HQ (Euc. B. 1. P. 20.); therefore Hm, or MA, is greater than $HQ - mQ$, or greater than $HQ - QS$, conſequently Q is not a point in the curve (Prop. 1.); therefore as PV cannot meet the curve at any other point, it muſt be a tangent.

Cor. 1. Hence, two right lines drawn from the foci to any point of the curve, make equal angles with the curve.

FIG. 22.

Cor. 2. If SP be produced to meet a line CE parallel to the tangent VP at P in E, the part PE is equal to AC. For draw HI parallel to CE, meeting SP produced in I. Then as PV is parallel to IH, the $\angle SPV = \angle PIH$, and $\angle HPV = \angle PHI$; but as the $\angle SPV = \angle HPV$, the $\angle PIH = \angle PHI$, therefore $PH = PI$, Alſo, as $SC = CH$, and CE is parallel to HI, $SE = IE$, or $IE = \frac{1}{2} SI$; and as $PI = PH$, $PI = \frac{1}{2}(PH + PI)$; therefore $PE = PI - IE = \frac{1}{2}(PH + PI - SI) = \frac{1}{2}(PH + PI - PS - PI) = \frac{1}{2}(PH - PS) = AC$ (Prop. 1.).

hence, $HP - PS = HA - AS = HA - MH = MA$, or $2AC$.

COR. 1. Hence, $SC = CH$.

COR. 2. By tranſpoſition, $HP = 2AC + SP$.

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PROP. IV.

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For take $Pm=PS$, and join Sm; then as $mP=PS$, $\angle mPn=\angle SPn$, and Pn is common to the triangles mPn, SPn, we have $mn=Sn$, and the angles at n right angles (Euc. B. 1. P. 4). Now if PV be not a tangent, let it cut the curve at ſome point Q, and join QS, Qm. Then as $mn=nS$, and the angles at n are right angles, and nQ common to the triangles mnQ, SnQ, we have $mQ=SQ$. Now (Prop. 1) $MA=HP-PS=HP-Pm=Hm$; alſo, $Hm+mQ$ is greater than HQ (Euc. B. 1. P. 20.); therefore Hm, or MA, is greater than $HQ-mQ$, or greater than $HQ-QS$, conſequently Q is not a point in the curve (Prop. 1.); therefore as PV cannot meet the curve at any other point, it muſt be a tangent.

Cor. 1. Hence, two right lines drawn from the foci to any point of the curve, make equal angles with the curve.

FIG. 22.

Cor. 2. If SP be produced to meet a line CE parallel to the tangent VP at P in E, the part PE is equal to AC. For draw HI parallel to CE, meeting SP produced in I. Then as PV is parallel to IH, the $\angle SPV=\angle PIH$, and $\angle HPV=\angle PHI$; but as the $\angle SPV=\angle HPV$, the $\angle PIH=\angle PHI$, therefore $PH=PI$. Alſo, as $SC=CH$, and CE is parallel to HI, $SE=IE$, or $IE=\frac{1}{2}SI$; and as $PI=PH$, $PI=\frac{1}{2}(PH+PI)$; therefore $PE=PI-IE=\frac{1}{2}(PH+PI-SI)=\frac{1}{2}(PH+PI-PS-PI)=\frac{1}{2}(PH-PS)=AC$ (Prop. 1.).

PROP. V.

If two perpendiculars be drawn from the foci to a tangent at any point of the hyperbola, they will meet it in the periphery of a circle deſcribed on the major axis as a diameter.

FIG. 23. For produce the perpendicular SY on the tangent PV, to meet HP in a; then the $\angle YPa = \angle YPS$ (Prop. 4.), $\angle SYP = \angle aYP$, and PY is common to the two triangles SPY, aPY, therefore $SY = Ya$, and $aP = SP$; hence, Ha $(= HP - Pa) = HP - SP =$ (Prop. 1.) AM. Now as $SC = CH$, and $SY = Ya$, CY is parallel to Ha; therefore $SH : SC :: Ha$ (AM) $: CY$; but $SC = \frac{1}{2} SH$; therefore $CY = \frac{1}{2} AM = AC$; hence, Y is in the periphery of a circle, whoſe center is C and radius CA. For the ſame reaſon, if PS be produced to meet HZ produced in b, $CZ = \frac{1}{2} Sb =$ (as Hb is parallel to Sa) $\frac{1}{2} Ha = AC$ as before.

PROP. VI.

The rectangle under the perpendiculars from the foci upon the tangent, is equal to the ſquare of the ſemi-axis minor.

Let HZ interſect the circle in p, and join Cp; then the $\angle pZY$ being a right angle, it muſt be in a ſemicircle, as Z and Y (Prop. 5.) are in the circumference; conſequently pCY muſt be a diameter;

hence, in the triangles HCp, $SC\Upsilon$, $HC=SC$, $Cp=C\Upsilon$, and the $\angle HCp=\angle SC\Upsilon$, therefore $Hp=S\Upsilon$; and by the property of the circle, $Hp \times HZ$, or $S\Upsilon \times HZ$, $=HA \times HM=BC^2$ (Prop. 2.).

Cor. 1. Since (Prop. 4.) $\angle SP\Upsilon=\angle HPZ$, the triangles $SP\Upsilon$, HPZ are similar, and $SP : S\Upsilon :: HP : HZ=\frac{S\Upsilon \times HP}{SP}$; hence, $\frac{S\Upsilon^2 \times HP}{SP}=HZ \times S\Upsilon=BC^2$; and $S\Upsilon^2=BC^2 \times \frac{SP}{HP}$.

Cor. 2. As BC is constant, $S\Upsilon$ varies as $\sqrt{\frac{SP}{HP}}$; but $HP = 2AC+SP$ (Cor. 2. Prop. 1.); therefore $S\Upsilon$ varies as $\sqrt{\frac{SP}{2AC+SP}}$, or $\frac{1}{S\Upsilon}$ varies as $\sqrt{\frac{2AC+SP}{SP}}$.

Cor. 3. Since $4S\Upsilon^2=\frac{4BC^2 \times SP}{2AC+SP}$, and (Prop. 3.) $4BC^2=2AC \times L$, we have $4S\Upsilon^2=\frac{2AC \times L \times SP}{2AC+SP}$; hence, $L \times SP : 4S\Upsilon^2 :: L \times SP : \frac{2AC \times L \times SP}{2AC+SP} :: 2AC+SP : 2AC$, and as $2AC+SP$ is greater than $2AC$, $L \times SP$ is greater than $4S\Upsilon^2$.

Prop.

PROP. VII.

If an ordinate be drawn to the major axis of an hyperbola, the product of the abſciſſas is to the ſquare of the ordinate, as the ſquare of the ſemi-axis major to the ſquare of the ſemi-axis minor.

FIG. 24. Draw PN perpendicular to AM, and take $AD=PS$; then,

$$PS=CD-CA$$

and $PH=CD+CA$ (Pr. 1.)

Alſo, $NS=NC-CS$

$NH=NC+CH=NC+CS$ (Pr. 1. Cor. 1.)

but $PH^2=PN^2+NH^2$ (Euc. B. 1. P. 47.)

$PS^2=PN^2+NS^2$; that is,

$$\overline{CD+CA}^2=PN^2+\overline{NC+CS}^2$$
$$\overline{CD-CA}^2=PN^2+\overline{NC-CS}^2$$

ſub. and $CD\times CA=NC\times CS$,

therefore $CD=\dfrac{NC\times CS}{CA}$; hence,

$$PS=CD-CA=\frac{NC\times CS}{CA}-AC=\frac{NC\times CS-CA^2}{CA}.$$

But $PS^2=PN^2+NS^2$; that is,

$$\frac{NC^2\times CS^2-2NC\times CS\times CA^2+CA^4}{CA^2}=PN^2+NC^2-2NC\times CS+CS^2,$$

which is the ſame equation as that for the ellipſe in Prop. 7. and conſequently there reſults from it the ſame concluſion, that is, $AN\times NM : PN^2 :: AC^2 : BC^2$.

COR. 1. Hence, $PN=\frac{BC}{AC}\times\sqrt{AN\times NM}$; therefore if upon the ſame axis major AM, to any other axis minor bg, an hyperbola vAw be drawn, and p be the point where it interſects the ordinate NP, then $Np=\frac{bC}{AC}\times\sqrt{AN\times NM}$; hence, $NP : Np$ as $BC : bC$, that is, in a given ratio; therefore for the ſame reaſon as in the parabola, Prop. 10, the area APS : area ApS in the ſame given ratio of $BC : bC$.

COR. 2. If $L=$ the latus rectum, and GB be the axis minor; then (Prop. 3.) $AM : BG :: BG : L$; hence, $AM : L :: AM^2 : BG^2 :: AC^2 : BC^2 :: AN\times NM : PN^2$, therefore $L=\frac{AM\times PN^2}{AN\times NM}=\frac{2AC\times PN^2}{AN\times NM}$.

COR. 3. If $AC=a$, $BC=b$, $CN=x$, $NP=y$, we have $\overline{x-a}\times\overline{x+a}$ $(x^2-a^2) : y^2 :: a^2 : b^2$, therefore $y^2=\frac{b^2}{a^2}\times\overline{x^2-a^2}$.

COR. 4. Becauſe $SP=CD-CA$, and $HP=CD+CA$, therefore $\frac{1}{2}.\overline{SP+HP}=CD$; hence, $CN\times CS$ $(=CA\times CD)=CA\times\frac{1}{2}.\overline{SP+HP}$.

COR. 5. As AC and BC are conſtant in the ſame hyperbola, $AN\times NM$ varies as PN^2.

PROP.

PROP. VIII.

If PT *be a tangent at* P, *then will* CN : CA :: CA : CT.

By Prop. 4. PT biſects the angle SPH;

$\therefore$ $HP : SP :: HT : TS$ (Euc. B. vi. P. 3.)
and $HP+PS : HP-PS :: HT+TS : HT-TS$,

but $HP-PS=2AC$ (Prop. 1.), and $HT-TS=HC+CT-TS=SC+CT-\overline{SC-CT}=2CT$; hence,

$$HP+PS : 2AC :: HS : 2CT,$$

or $AC : CT :: \frac{1}{2}.\overline{HP+PS} : \frac{1}{2}.HS$, or CS,
$:: AC\times\frac{1}{2}.\overline{HP+PS} : CS\times AC$
$:: CS\times CN$ (Pr. 7. Cor. 4.) $: CS\times AC$
$:: CN : AC$,

hence, $CN : CA :: CA : CT$.

COR. 1. By diviſion, $NA : CA :: AT : CT$, and as CA is greater than CT, NA is greater than AT.

COR. 2. If $AC=a$, $BC=b$, $CN=x$, $NP=y$, then $x : a :: a : CT=\frac{a^2}{x}$; hence, $NT=x-\frac{a^2}{x}=\frac{x^2-a^2}{x}$; conſequently (Cor. 3. Prop. 7.) $PN^2 : TN^2 :: \frac{b^2}{a^2}\times\overline{x^2-a^2} : \frac{\overline{x^2-a^2}|^2}{x^2} :: b^2x^2 : a^2x^2-a^4$.

PROP.

PROP. IX.

To draw an aſymptote to an hyperbola.

By Cor. 2. Prop. 8. $CT=\frac{a^2}{x}$; now increaſe x without limit, and CT will be diminiſhed without limit, conſequently T approaches C as it's limit; therefore (Def. 9. Cor.) the aſymptote paſſes through C. Alſo (Cor. 2. Prop. 8.), $PN^2 : TN^2 :: b^2x^2 : a^2x^2-a^4$, the limit of which ratio (by increaſing x without limit) is $b^2x^2 : a^2x^2 :: b^2 : a^2$, and $PN : TN :: b : a$, or, as T now coincides with C, $PN : CN :: b : a$. But as P approaches the aſymptote as it's limit, and T approaches C as it's limit, the limit of the triangle PNT is a triangle ſimilar to aAC; therefore, at that limit, $PN : CN :: Aa : AC\ (a)$; hence, and the laſt proportion, we get $Aa=b=BC$; if therefore Aa be erected perpendicular to AC, and equal to BC, and CaZ be drawn, it will be the aſymptote. FIG. 26.

Hence, if on the other ſide of AC, a perpendicular Ab be erected equal to BC, and CbL be drawn, it will be an aſymptote to the other part; and if theſe aſymptotes be produced on the other ſide of the center, they will manifeſtly be aſymptotes to the oppoſite hyperbola.

COR. 1. Theſe aſymptotes will be aſymptotes to the conjugate hyperbolas; for perpendiculars to BG, at B and G, equal to CA, will terminate at a and b, and therefore, by the propoſition, CaZ, CbL will be aſymptotes to theſe hyperbolas.

Cor. 2. The aſymptotes make equal angles with the axis major.

Prop. X.

FIG. 27. *If the ordinate* PN *to the axis, be produced to meet the aſymptotes in* Q *and* q, *then* $PQ \times Pq = Aa^2$.

For by ſimilar triangles CAa, CNQ,

$CN^2 : CA^2 :: NQ^2 : Aa^2$;
but $CA^2 : CN^2 - CA^2 :: Aa^2 : PN^2$ (Pr. 7. Cor. 3.)
$\therefore CN^2 : CN^2 - CA^2 :: NQ^2 : PN^2$
div. $CN^2 : CA^2 :: NQ^2 : NQ^2 - PN^2$
$:: NQ^2 : \overline{NQ + PN} \times \overline{NQ - PN}$
$:: NQ^2 : Pq \times PQ$

From which, and the firſt proportion, we have $Pq \times PQ = Aa^2$.

Cor. 1. For the ſame reaſon, $pQ \times pq = Ab^2$, or Aa^2; therefore $PQ \times Pq = pq \times pQ$.

Cor. 2. Hence, if $WVvw$ be drawn parallel to Qq, $VW \times Vw = Aa^2$; therefore $VW \times Vw = PQ \times Pq$.

Prop. XI.

FIG. 28. ***If a line* XVpx *be drawn in any poſition, cutting the two aſymptotes in* X, x, *and the curve in* V, p, *then will* VX × Vx = pX × px.**

For through V and P draw two lines $WVvw$, $\mathcal{Q}Pq$, perpendicular to the axis, and through P draw RPr parallel to Xx; then on account of the parallel lines Ww, $\mathcal{Q}q$, and Xx, Rr, the triangles $\mathcal{Q}PR$ and XVW, Pqr and Vxw are ſimilar, therefore

$$\begin{array}{l} VX : VW :: PR : P\mathcal{Q} \\ Vx : Vw :: Pr : Pq \\ \hline \therefore VX \times Vx : VW \times Vw :: PR \times Pr : P\mathcal{Q} \times Pq; \end{array}$$

but (Cor. 2. Prop. 10.) $VW \times Vw = P\mathcal{Q} \times Pq$; therefore $VX \times Vx = PR \times Pr$. Conceive now the line RPr to move parallel to itſelf, until it becomes a tangent LEM at the point E, then PR becomes EL, and Pr becomes EM; hence, $VX \times Vx = EL \times EM$. For the ſame reaſon, $pX \times px = EM \times EL$; therefore $VX \times Vx = pX \times px$.

Cor. 1. Hence, $VX = px$. For $VX \times Vx = pX \times px$, that is, $VX \times \overline{Vp + px} = px \times \overline{Vp + VX}$; take $VX \times px$ from both ſides, and we have $VX \times Vp = px \times Vp$, therefore $VX = px$.

Cor. 2. Hence, if we ſuppoſe Xx to move parallel to itſelf and coincide with LM, we ſhall have $LE = EM$.

Cor. 3. Hence, $VX \times Vx = EL^2$.

Cor. 4. As $VX = px$, therefore $Xp = Vx$; conſequently $VX \times Xp = EL^2$.

Cor. 5. Let Xx cut CE produced in O; then as the lines LM, Xx are parallel, and $LE = EM$, we have $XO = xO$; but (Cor. 1.) $VX = px$; therefore $VO = pO$; hence, the diameter CEZ biſects all it's ordinates.

PROP. XII.

If BCb *be drawn parallel to* LM, *and from* E, EB, Eb, *be drawn parallel to* CM, CL *reſpectively, then* BC=bC=LE=ME.

For as CM is parallel to BE, and BC to EM, $CBEM$ is a parallelogram, therefore $BC=EM$. For the ſame reaſon, $bC=EL$; but $EL=EM$; therefore $BC=bC=LE=ME$.

COR. 1. As (Cor. 3. Prop. 11.) $VX \times Vx = EL^2$, therefore $VX \times Vx = BC^2$.

COR. 2. As Ca is parallel to BE, and $BC=bC$, therefore $Ea=ab$.

DEF. 11. Bb is called a *conjugate* diameter to the diameter CEZ. Theſe are alſo called *Conjugate Diameters.*

PROP. XIII.

If CE *be produced to meet the oppoſite hyperbola in* D, *then* $DO \times EO : OV^2 :: CE^2 : CB^2$.

For by ſimilar triangles CEL, COX, $CE : EL :: CO : OX = \frac{EL \times CO}{CE} = Qx$ (Prop. 11. Cor. 5.); hence, $VX = OX - OV = \frac{EL \times CO}{CE} - OV = \frac{EL \times CO - CE \times OV}{CE}$, and $Vx = Ox + OV = \frac{EL \times CO}{CE} + OV = \frac{EL \times CO + CE \times OV}{CE}$; therefore (Cor. 1.

Prop. 12.) $\frac{EL^2 \times CO^2 - CE^2 \times OV^2}{CE^2} = CB^2$; hence, $CB^2 \times CE^2 =$ (as $EL = CB$) $CB^2 \times CO^2 - CE^2 \times OV^2$, therefore $CE^2 \times OV^2 = CB^2 \times \overline{CO^2 - CE^2} = CB^2 \times \overline{CO + CE} \times \overline{CO - CE} = CB^2 \times \overline{CO + CD}\ (DO) \times EO$; hence, $DO \times EO : OV^2 :: CE^2 : CB^2$.

PROP. XIV.

If DV, SE, *be drawn parallel to the aſymptote* YC, *and* ET *parallel to* CZ, *then* DV × DC = SE × SC. FIG. 29.

For draw VR parallel to CZ, then the triangles XVD, LES, VxR, EMT, being ſimilar, we have

$$\begin{array}{l} DV : SE :: VX : LE \\ VR\ (DC) : ET\ (SC) :: Vx : EM\ (LE) \\ \hline \therefore DV \times DC : SE \times SC :: VX \times Vx : LE^2; \end{array}$$

but (Cor. 3. Prop. 11.) $VX \times Vx = LE^2$; therefore $DV \times DC = SE \times SC$.

COR. 1. Becauſe $LE = EM$ (Cor. 2. Prop. 11.), and SE is parallel to CM, therefore $LS = SC$, or $SC = \frac{1}{2} CL$.

COR. 2. Hence, $DV \times DC$ is a conſtant quantity, and therefore DV varies inverſely as DC.

COR. 3. If the point E coincide with A, the angle $ACT = \angle ACS$ (Cor. 2. Prop. 9.); but as $ATCS$ is a parallelogram, the $\angle ACT = \angle CAS$; therefore the $\angle CAS = \angle ACS$; conſequently $AS = SC = AT$. FIG. 30.

COR. 4. Hence, $DV \times DC = AS^2$.

Cor. 5. The triangles BCS, CGT are ſimilar, and $BC=CG$; therefore $GT=CS=TA$. For the ſame reaſon, $BS=SA$.

Prop. XV.

If LMN *be drawn parallel to CD, then* LM=MN.

For (Cor. 4. Prop. 14.) $LM \times MC = TA^2$, and $NM \times MC = TG^2$; but $TA=TG$ (Cor. 5. Pr. 14.); therefore $LM=MN$.

Cor. Hence (Prop. 12. Cor. 2.), NCa is a conjugate diameter to the diameter LCb, and conſequently parallel to a tangent at L. For the ſame reaſon, LCb is a conjugate diameter to the diameter NCa, and therefore parallel to a tangent at N.

•

Prop. XVI.

All the parallelograms circumſcribing any two conjugate diameters of an hyperbola, are equal.

FIG. 31. Let $abcd$ be a parallelogram circumſcribing the two conjugate diameters DK, PG; join PD, and draw Dn perpendicular to the aſymptote CZ. Now (Prop. 15. and it's Cor.) the aſymptote CZ biſects DP, which is the diagonal of the parallelogram $CDaP$; therefore the angle a muſt be ſituated in that aſymptote. Alſo (Prop. 15.), DmP (m being the point where DP interſects CZ) is parallel to the aſymptote CL, and conſequently given in poſition, or the angle Dmn is conſtant, and therefore Dm is to Dn

Dn in a given ratio; but (Prop. 14. Cor. 2.) Dm varies inverſely as Cm; therefore Dn varies inverſely as Cm; but (Cor. 1. Prop. 14.) $Cm = \frac{1}{2} Ca$; therefore Dn varies inverſely as Ca, and conſequently $Dn \times Ca$ is conſtant; but $Dn \times Ca$ is equal to the area of the parallelogram $DaPC$; therefore the parallelogram $DaPC$ is a conſtant quantity; but the parallelogram $DaPC$ is a fourth part of the parallelogram $abcd$; therefore the parallelogram $abcd$ is a conſtant quantity.

Cor. 1. Hence, the parallelogram $abcd =$ the parallelogram $vwxy$ deſcribed about the major and minor axes AM, BE; therefore their fourth parts, $aDCP$, $AvBC$, are equal; if therefore PF be perpendicular to CD, we have $CD \times PF = AC \times BC$.

Cor. 2. Becauſe the triangles SPY, HPZ, FPE are ſimilar, we have FIG. 22.

$$\begin{array}{l} SP : SY :: PE : PF \\ HP : HZ :: PE : PF \\ \hline \therefore\ SP \times HP : SY \times HZ :: PE^2 : PF^2; \end{array}$$

but (Prop. 6.) $SY \times HZ = BC^2$; and (Cor. 2. Pr. 4.) $PE^2 = AC^2$; therefore $SP \times HP = \frac{BC^2 \times AC^2}{PF^2} = CD^2$ by the laſt Cor.

PROP. XVII.

The difference of the ſquares of any two ſemi-conjugate diameters, is equal to the difference of the ſquares of the two ſemi-axes.

FIG. 31. Draw PN perpendicular to the axis HS; then (Euc. B. II. P. 12. 13.)

$$PS^2 = PC^2 + CS^2 - 2CS \times CN$$
$$PH^2 = PC^2 + CS^2\ (CH^2) + 2CS\ (CH) \times CN$$

$$\therefore\ PS^2 + PH^2 = 2PC^2 + 2CS^2$$

and $2PC^2 = PS^2 + PH^2 - 2CS^2$;
but $2CD^2 = 2PS \times PH$ (Prop. 16. Cor. 2.)

$\therefore\ 2PC^2 - 2CD^2 = \overline{PH - PS}^2 - 2CS^2 =$ (Prop. 1.)

$4AC^2 - 2CS^2 =$ (Def. 4.) $4AC^2 - 2AB^2 = 4AC^2 - 2AC^2 - 2BC^2 = 2AC^2 - 2BC^2$; hence, $PC^2 - CD^2 = AC^2 - BC^2$.

PROP. XVIII.

FIG. 15. *If* ASW *be the axis of the hyperbola* PAL, *the distance*

$$\mathrm{SP} = \frac{\mathrm{BC}^2}{\mathrm{CM} - \mathrm{SC} \times \mathit{cos.}\ \mathrm{PSW}}.$$

Let PN be an ordinate to MAN; then (Prop. 7.) $CM^2 : CB^2 :: AN \times NM : PN^2$; but $CB^2 = AB^2 - AC^2 = SC^2 - CM^2$ (Def. 4.), and $AN \times NM = \overline{CN - CA} \times \overline{CN + CM} = \overline{CN - CM} \times \overline{CN + CM} = CN^2 - CM^2 = \overline{SC \mp SN}^2 - CM^2$; hence, $CM^2 : SC^2 - CM^2 :: \overline{SC \mp SN}^2 - CM^2 : PN^2 = \dfrac{\overline{SC^2 - CM^2} \times \overline{\overline{SC \mp SN}^2 - CM^2}}{CM^2} =$

$$\frac{SC^4 \pm SC^2 \times SN^2 \mp 2SC^3 \times SN - 2SC^2 \times CM^2 \pm 2SC \times SN \times CM^2 - CM^2 \times SN^2 + CM^4}{CM^2};$$

add SN^2 to both sides of the equation, and we have $SP^2\ (PN^2 + SN^2) =$

SC^4

$$\frac{SC^4 + SC^2 \times SN^2 \mp 2SC^3 \times SN - 2SC^2 \times CM^2 \pm 2SC \times SN \times CM^2 + CM^4}{CM^2},$$

the ſquare root of which is $SP = \dfrac{SC^2 \mp SN \times SC - CM^2}{CM}$ $=$ (as $SC^2 - CM^2 = BC^2$) $\dfrac{BC^2 \mp SN \times SC}{CM}$; but $\mp SN = SP \times \text{coſ. } PSW$; therefore $SP = \dfrac{BC^2 + SC \times SP \times \text{coſ. } PSW}{CM}$; hence, $SP = \dfrac{BC^2}{CM - SC \times \text{coſ. } PSW}$.

Prop. XIX

If a line be drawn from the focus to any point of the curve, and from that point a line be drawn perpendicular to the directrix, theſe lines are in a given ratio.

For $CV = \dfrac{CA^2}{SC}$ (Prop. 8.), therefore $SV = SC - \dfrac{CA^2}{SC} = \dfrac{SC^2 - CA^2}{SC} = \dfrac{AB^2 - CA^2}{SC} = \dfrac{BC^2}{SC}$; and by Trigonometry, rad. $= 1$: coſ. PSW :: SP, or $\dfrac{BC^2}{CM - SC \times \text{coſ. } PSW}$ (Prop. 18.), : $SN = \dfrac{BC^2 \times \text{coſ. } PSW}{CM - SC \times \text{coſ. } PSW}$; therefore $KP = VN = SV + SN = \dfrac{BC^2}{SC} + \dfrac{BC^2 \times \text{coſ. } PSW}{CM - SC \times \text{coſ. } PSW} = \dfrac{CM \times BC^2}{SC \times \overline{CM - SC \times \text{coſ. } PSW}}$; hence, $SP : PK ::$

FIG. 31. Draw PN perpendicular to the axis HS; then (Euc. B. 11. P. 12. 13.)

$$\begin{array}{l} PS^2 = PC^2 + CS^2 - 2CS \times CN \\ \underline{PH^2 = PC^2 + CS^2\ (CH^2) + 2CS\ (CH) \times CN} \end{array}$$

$\therefore\ PS^2 + PH^2 = 2PC^2 + 2CS^2$

and $2PC^2 = PS^2 + PH^2 - 2CS^2$;

but $\underline{2CD^2 = 2PS \times PH}$ (Prop. 16. Cor. 2.)

$\therefore\ 2PC^2 - 2CD^2 = \overline{PH - PS}^2 - 2CS^2 =$ (Prop. 1.)

$4AC^2 - 2CS^2 =$ (Def. 4.) $4AC^2 - 2AB^2 = 4AC^2 - 2AC^2 - 2BC^2 = 2AC^2 - 2BC^2$; hence, $PC^2 - CD^2 = AC^2 - BC^2$.

PROP. XVIII.

FIG. 15. *If* ASW *be the axis of the hyperbola* PAL, *the diſtance* $\mathrm{SP} = \dfrac{\mathrm{BC}^2}{\mathrm{CM} - \mathrm{SC} \times \mathit{cos.}\ \mathrm{PSW}}$.

Let PN be an ordinate to MAN; then (Prop. 7.) $CM^2 : CB^2 :: AN \times NM : PN^2$; but $CB^2 = AB^2 - AC^2 = SC^2 - CM^2$ (Def. 4.), and $AN \times NM = \overline{CN - CA} \times \overline{CN + CM} = \overline{CN - CM} \times \overline{CN + CM} = CN^2 - CM^2 = \overline{SC \mp SN}^2 - CM^2$; hence, $CM^2 : SC^2 - CM^2 :: \overline{SC \mp SN}^2 - CM^2 : PN^2 = \dfrac{\overline{SC^2 - CM^2} \times \overline{\overline{SC \mp SN}^2 - CM^2}}{CM^2} =$

$$\frac{SC^4 + SC^2 \times SN^2 \mp 2SC^3 \times SN - 2SC^2 \times CM^2 \pm 2SC \times SN \times CM^2 - CM^2 \times SN^2 + CM^4}{CM^2};$$

add SN^2 to both ſides of the equation, and we have $SP^2\ (PN^2 + SN^2) =$

SC^4

$$\frac{SC^4 + SC^2 \times SN^2 \mp 2SC^3 \times SN - 2SC^2 \times CM^2 \pm 2SC \times SN \times CM^2 + CM^4}{CM^2},$$

the ſquare root of which is $SP = \frac{SC^2 \mp SN \times SC - CM^2}{CM}$ $= (\text{as } SC^2 - CM^2 = BC^2)\ \frac{BC^2 \mp SN \times SC}{CM}$; but $\mp SN = SP \times \text{coſ. } PSW$; therefore $SP = \frac{BC^2 + SC \times SP \times \text{coſ. } PSW}{CM}$; hence, $SP = \frac{BC^2}{CM - SC \times \text{coſ. } PSW}$.

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If a line be drawn from the focus to any point of the curve, and from that point a line be drawn perpendicular to the directrix, theſe lines are in a given ratio.

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$\frac{BC^2}{\overline{CM - SC \times \text{coſ. } PSW}} : \frac{CM \times BC^2}{SC \times \overline{CM - SC \times \text{coſ. } PSW}}$ :: $SC : CM$ a conſtant ratio.

Cor. Hence, $SP : PK :: SA : AV$; and as SA is greater than AV (Prop. 8. Cor. 1.), SP is greater than PK.

Prop. XX.

To determine the diameter of the circle of curvature at any point of an hyperbola, and the chords thereof which paſs through the center and focus of the hyperbola.

FIG. 32. Let $PaLV$ be the circle of curvature at the point P, $\mathcal{Q}v$ an ordinate to the diameter PG, and $\mathcal{Q}R$ parallel to Pv; then (Prop. 13.), Pv ($\mathcal{Q}R$) $\times vG : \mathcal{Q}v^2 :: PC^2 : CD^2$, therefore $\frac{\mathcal{Q}v^2}{\mathcal{Q}R} = \frac{CD^2 \times vG}{PC^2} =$ (if $\mathcal{Q}$ move to P, in the ultimate ſtate of the arc $\mathcal{Q}P$) $\frac{CD^2 \times 2PC}{PC^2} = \frac{2CD^2}{PC}$. But (Newton's *Prin.* Lib. 1. Lem. 7. Cor.) the limit of the ratio of $\mathcal{Q}v : \mathcal{Q}P$ is a ratio of equality; therefore $\frac{2CD^2}{PC} = \frac{\mathcal{Q}P^2}{\mathcal{Q}R}$ ultimately; hence (Lem. pag. 7.), $\frac{2CD^2}{PC} = PV$ the *chord of curvature* paſſing through the *center* C. Alſo, if LPF be perpendicular to PR, or CD, it muſt paſs through the center of the circle; hence, by ſimilar triangles PVL, PCF, $PF : PC :: PV \left(\frac{2CD^2}{PC}\right) : PL = \frac{2CD^2}{PF} =$ the *diameter* of curvature. Laſtly, if PS (produced if neceſſary) interſect the circle in a, and La be joined, by

ſim. tri. PEF, PLa, PE, or AC, (Cor. 2. Pr. 4.) : PF :: $PL\left(=\frac{2CD^2}{PF}\right) : Pa = \frac{2CD^2}{AC}$ = the *chord of curvature* paſſing through the *focus*.

PROP. XXI.

If a cone RLW *be cut through the ſide* RL *by a plane* AFG, *which being produced will meet the other ſide of the cone* WR *produced in any point* M, *the ſection will be an hyperbola.* FIG. 33.

Let $XPYV$ be a ſection of the cone parallel to the baſe LW, which being a circle, $NX \times NY = NP^2$; draw RQ parallel to the diameter YX, and the triangles RMQ, NMY, and AQR, AXN will be ſimilar; hence,

$$\begin{array}{l} AQ : QR :: AN : NX \\ MQ : QR :: NM : NY, \\ \hline \therefore AQ \times MQ : QR^2 :: AN \times NM : NX \times NY, \text{ or } NP^2; \end{array}$$

but the two firſt terms are conſtant quantities; therefore $AN \times NM$ is to PN^2 in a conſtant ratio; conſequently (Cor. 5. Prop. 7.) the ſection is an hyperbola, whoſe major axis is AM.

Hence, according to our definition, the *Hyperbola* is a conic ſection.

PROP.

PROP. XXII.

Given the diſtance of any point of a conic ſection from the focus, the latus rectum, and the poſition of the tangent, to determine the conic ſection.

FIG. 10, 23. The diſtance SP, and the poſition of the tangent being given, the perpendicular SY will be given; now (Cor. 1. Prop. 6. of the *Ellipſe* and *Hyperbola*), $\frac{SY^2 \times HP}{SP} = BC^2$; alſo, $2AC = HP \pm SP$, according as the curve is an ellipſe or hyperbola; therefore the latus rectum $L \left(= \frac{4BC^2}{2AC}\right) = \frac{4SY^2 \times HP}{SP \times \overline{HP \pm SP}}$, conſequently $HP = \frac{\pm L \times SP^2}{4SY^2 - L \times SP}$.

Now in the *Ellipſe* (Cor. 3. Prop. 6.), $L \times SP$ is leſs than $4SY^2$; when therefore $L \times SP$ is leſs than FIG. 10. $4SY^2$, draw PH on the ſame ſide of the tangent with SP, making an angle $HPZ = \angle SPY$, and take $HP = \frac{L \times SP^2}{4SY^2 - L \times SP}$, and H will be the other focus; alſo, $SP + PH =$ the major axis; and as HP is known, HZ will be given; conſequently $SY \times HZ$, or BC^2, will be known, and therefore the ellipſe is determined.

In the *Hyperbola* (Cor. 3. Prop. 6.), $L \times SP$ is greater than $4SY^2$; when therefore $L \times SP$ is greater FIG. 23. than $4SY^2$, draw PH on the other ſide of the tangent in reſpect to SP, making an angle $HPZ = \angle SPY$, and take $HP = \frac{-L \times SP^2}{4SY^2 - L \times SP}$, or $\frac{L \times SP^2}{L \times SP - 4SY^2}$, and H will

H will be the other focus; alſo, $HP - SP =$ the major axis; and as HP is known, HZ will be given; conſequently $SY \times HZ$, or BC^2, will be known, and therefore the hyperbola is determined.

In the *Parabola* (Cor. 3. Prop. 8.) $L \times SP = 4SY^2$, and in this caſe we have the following conſtruction. Make the angle $TPM = \angle TPS$, and take $PM = PS$, draw DME perpendicular to PM, through S draw ZSL parallel to PM, and biſect SL in A, and A will be the vertex of the parabola, from whence the parabola is determined. FIG. 1.

Prop. XXIII.

Let S *be the focus of a conic ſection,* SR, SP, SQ, *three lines given in length and poſition; to deſcribe a conic ſection through the three given points* R, P, Q. FIG. 12.

Join PQ, PR, and produce them to a and b, ſo that $Qa : Pa :: QS : PS$, and $Pb : Rb :: PS : RS$, and through ab draw xy, perpendicular to which draw SV, QK, PL, RO; and in VS indefinitely produced, take $VA : AS$, and $VM : MS$, in the ratio of $QK : QS$, and MA is the major axis of the figure, which will be an ellipſe, parabola, or hyperbola, according as QR is greater, equal to, or leſs than QS; this appears by Prop. 17. Cor. of the *Ellipſe*, Def. 1. of the *Parabola*, and Prop. 19. Cor. of the *Hyperbola:* and in the firſt caſe, M lies the ſame way from V that A does; in the ſecond caſe, M goes off in infinitum; and in the third caſe, M lies on the other ſide of V. For $QS : PS :: Qa : Pa ::$ (by ſim. tri.) $QK : PL$;

hence, $PS : PL :: QS : QK :: SA : AV :: SM : MV$. Alſo, $PS : RS :: Pb : Rb :: PL : RO$; hence, $RS : RO :: PS : PL :: QS : QK :: SA : AV :: SM : MV$. Therefore (by the propoſitions quoted above) R, P, Q lie in a conic ſection, whoſe focus is S, and major axis AM.

By this propoſition, the orbit of a planet may be found, having given three diſtances from the ſun, and the angles between them.

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hence, $PS : PL :: QS : QK :: SA : AV :: SM : MV$. Alſo, $PS : RS :: Pb : Rb :: PL : RO$; hence, $RS : RO :: PS : PL :: QS : QK :: SA : AV :: SM : MV$. Therefore (by the propoſitions quoted above) R, P, Q lie in a conic ſection, whoſe focus is S, and major axis AM.

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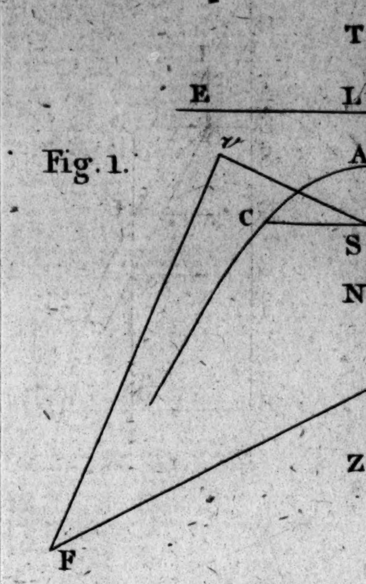


Fig. 1.

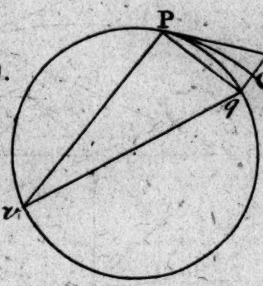
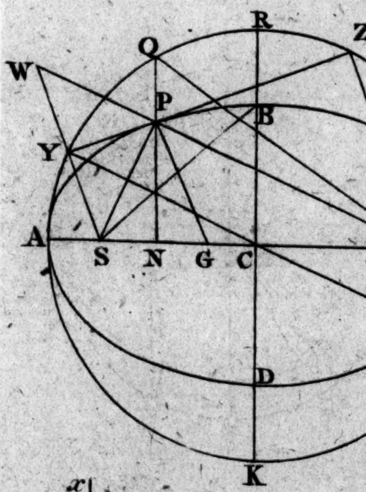


Fig. 4.



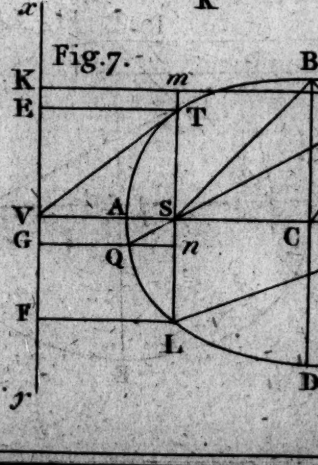
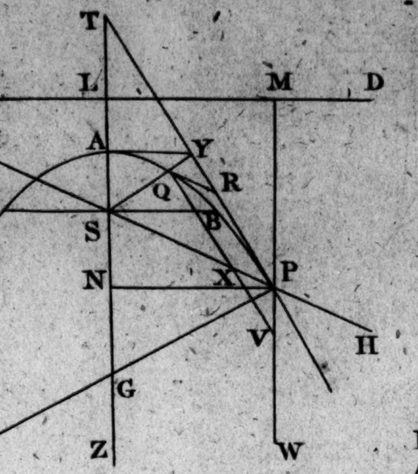


Fig. 7.



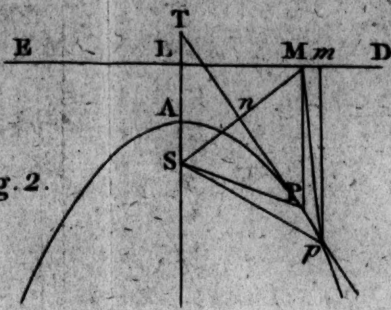
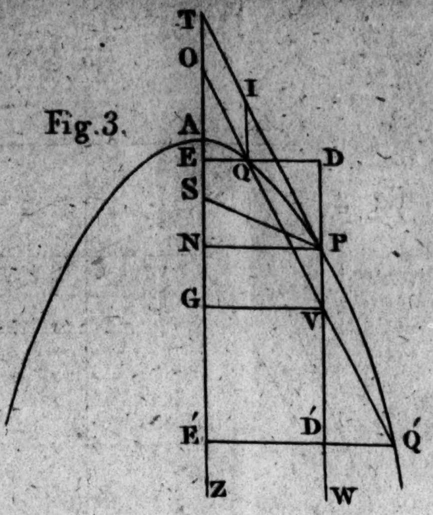


Fig. 2.



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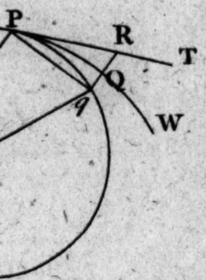


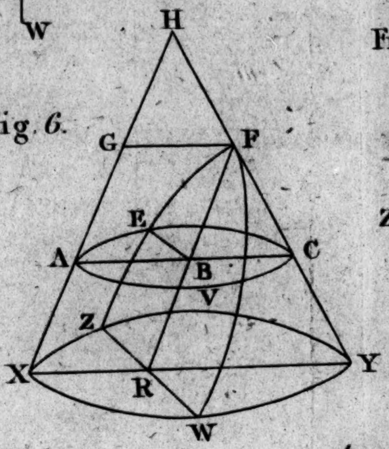


Fig. 6.

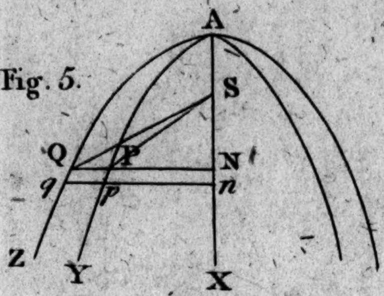
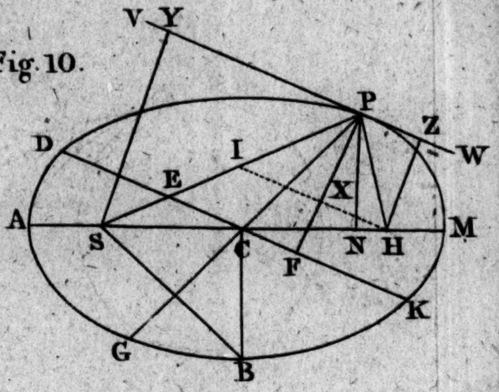


Fig. 5.



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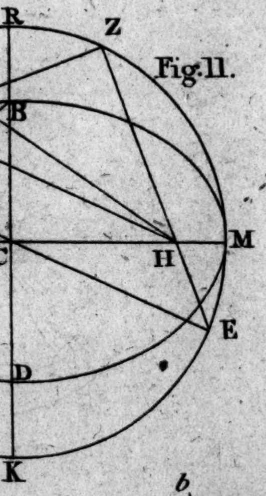
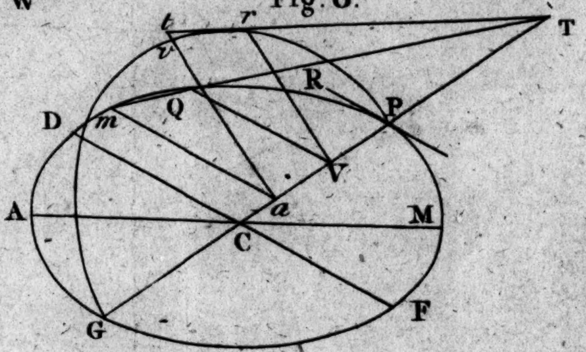


Fig. 11.



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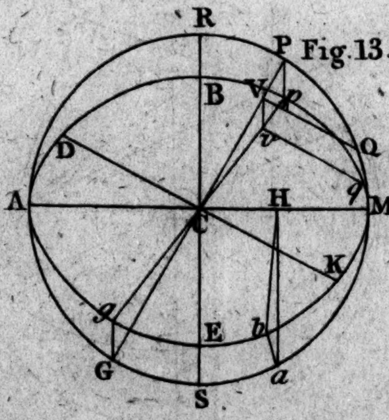
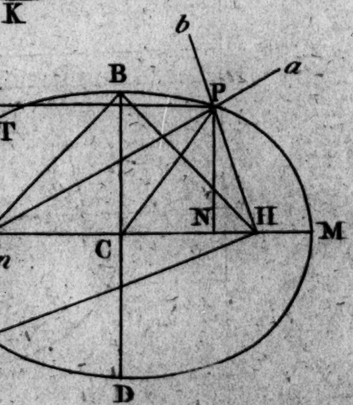
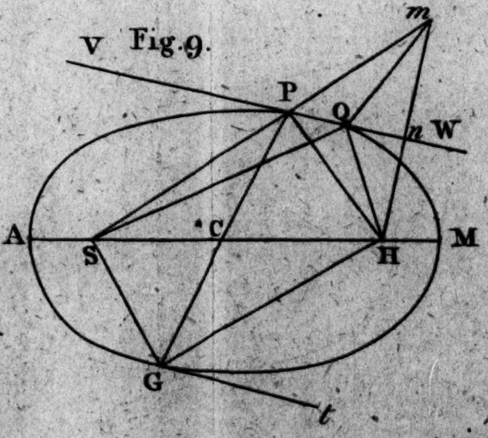


Fig. 13.





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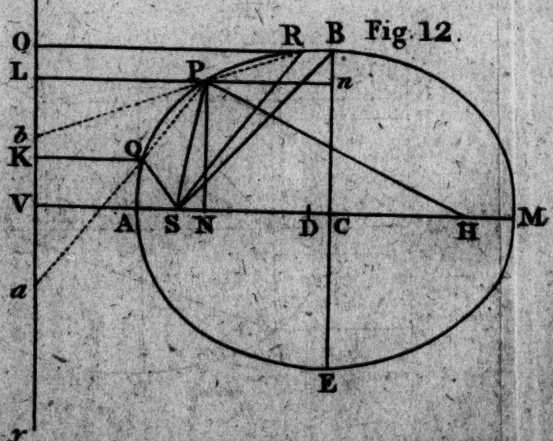


Fig. 12.

Pl. II.

Fig. 14.

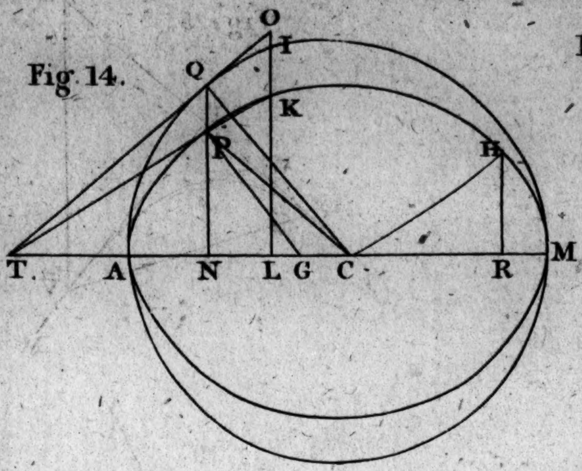


Fig. 17.

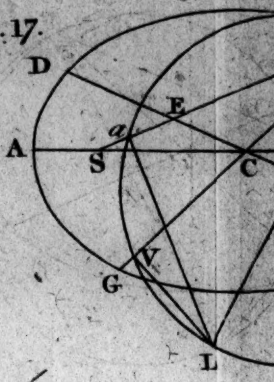


Fig. 15.

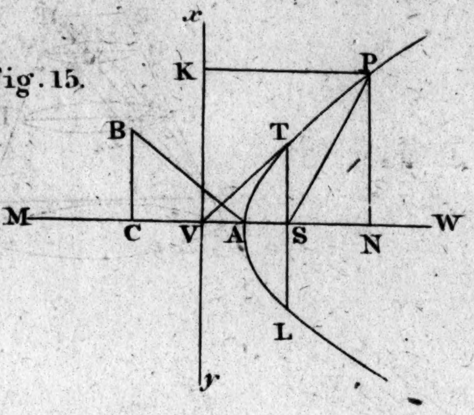


Fig. 16.

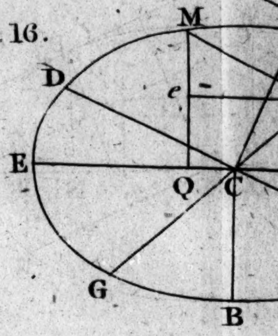
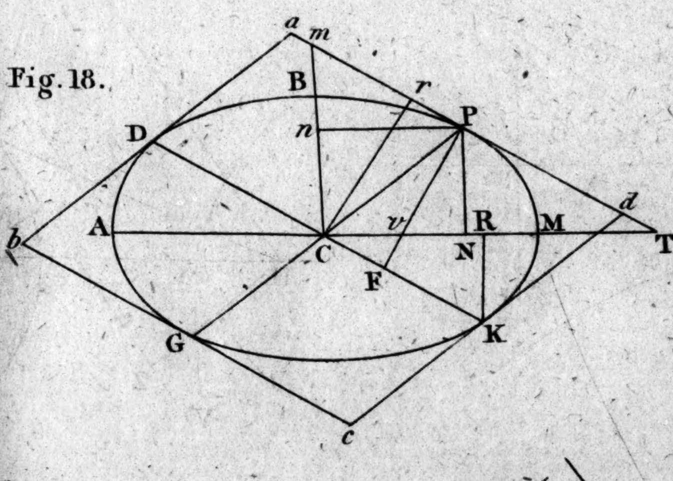


Fig. 18.



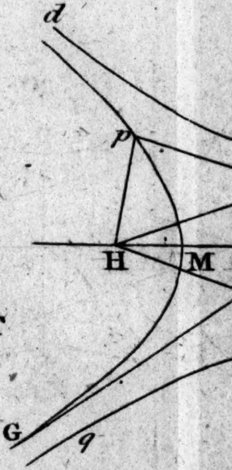


Fig. 21.

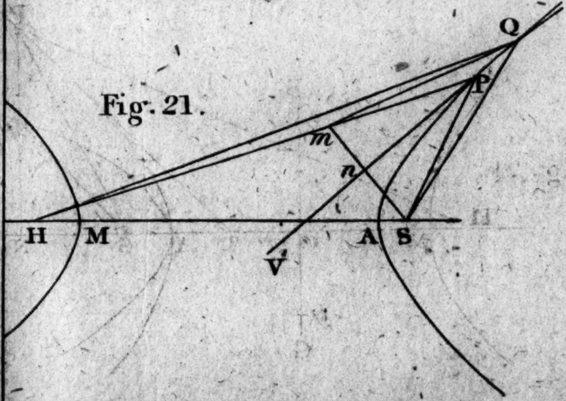
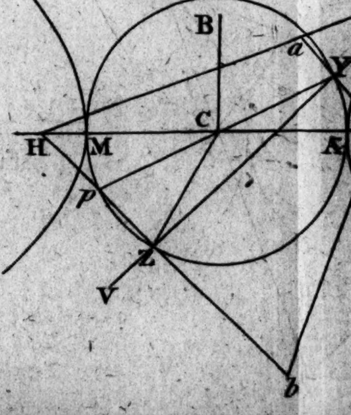


Fig. 23.



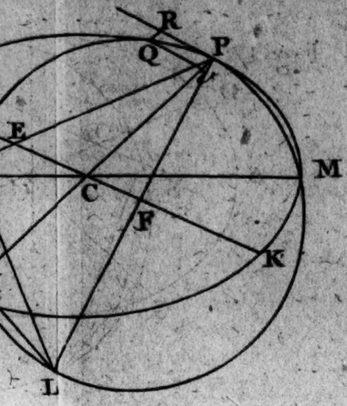
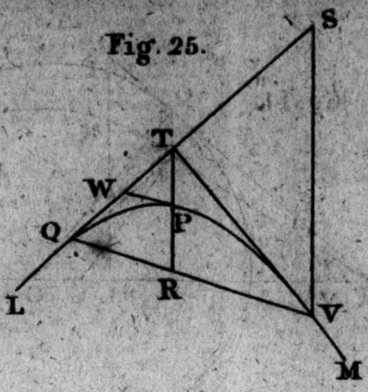
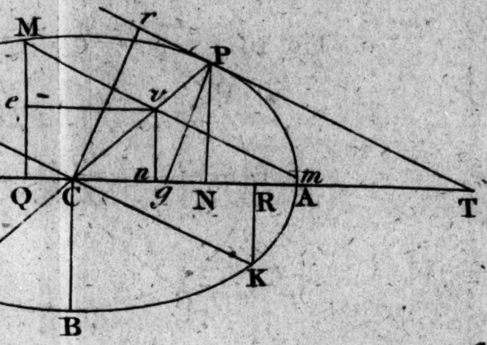


Fig. 25.





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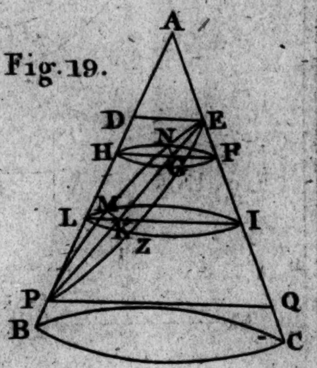

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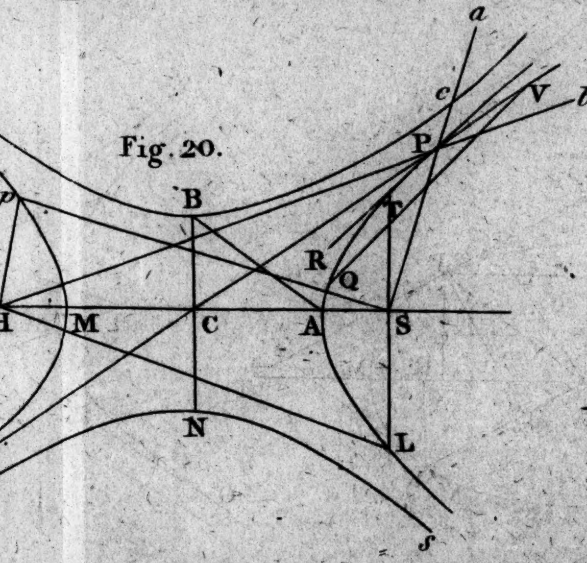

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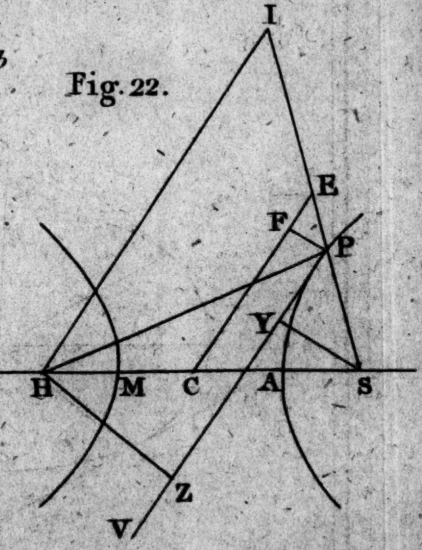

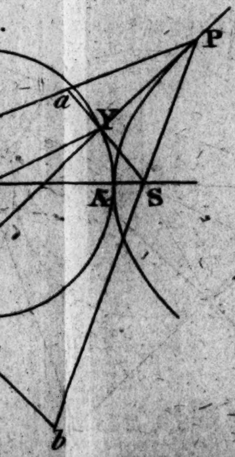

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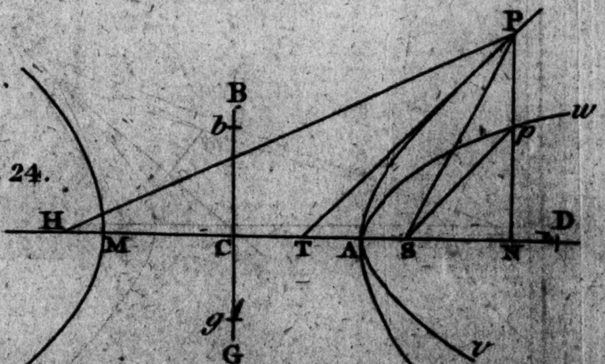


Pl. III.

Fig. 26.

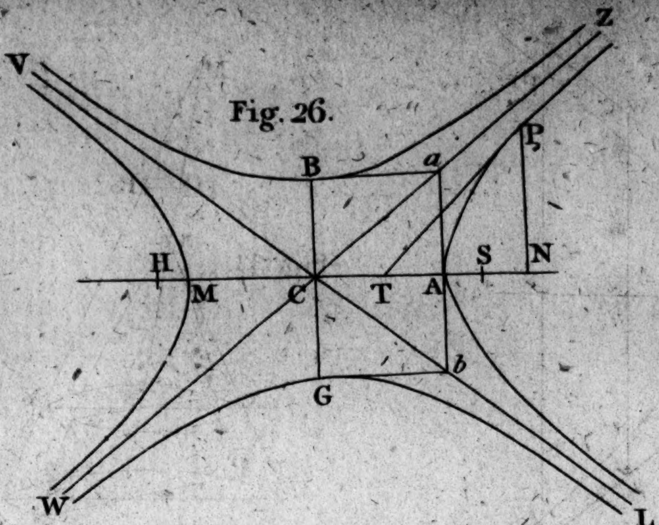


Fig. 27

Fig. 29.

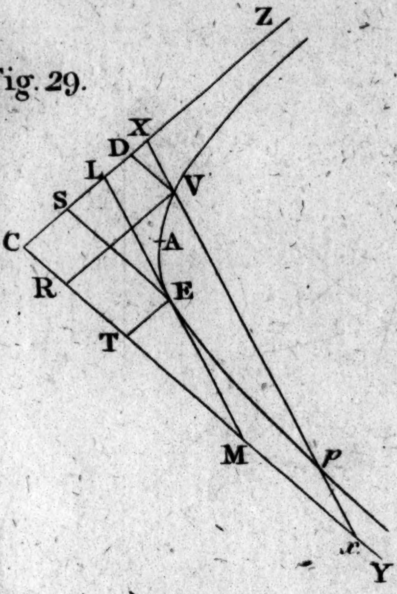


Fig. 30.

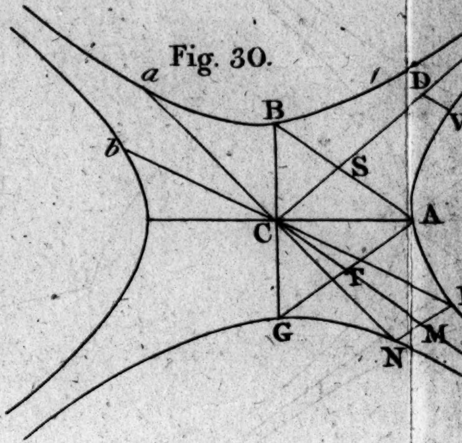
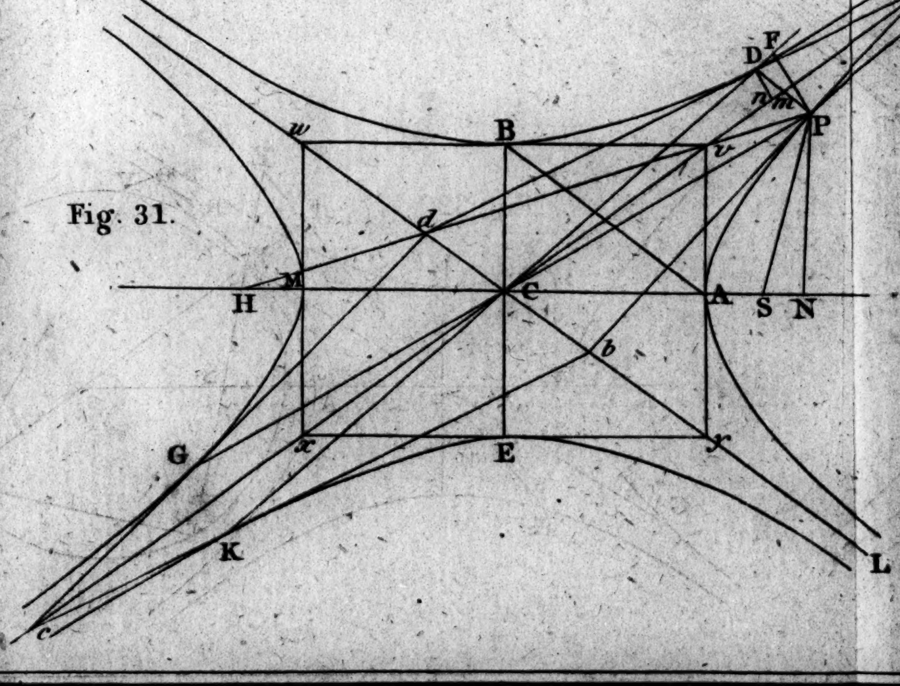
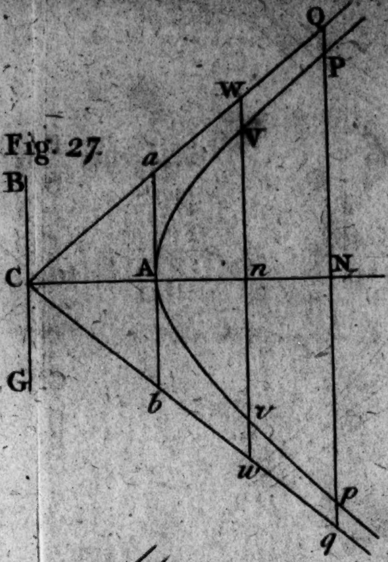


Fig. 31.

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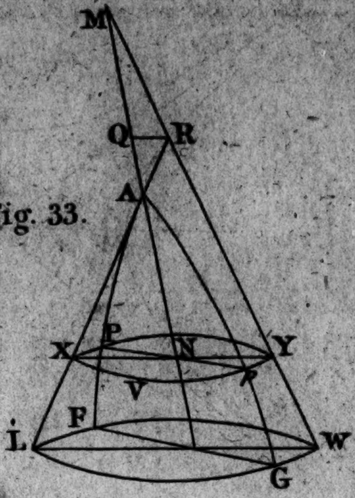
Fig. 33.

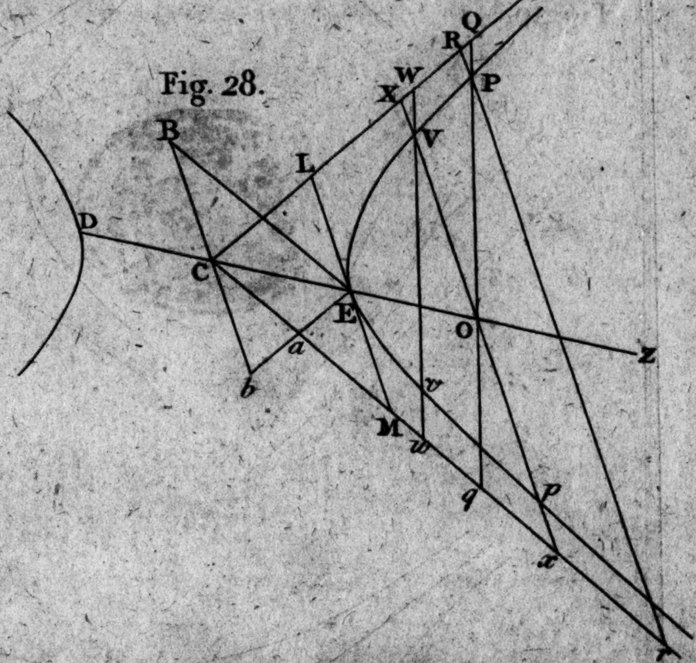
Fig. 28.

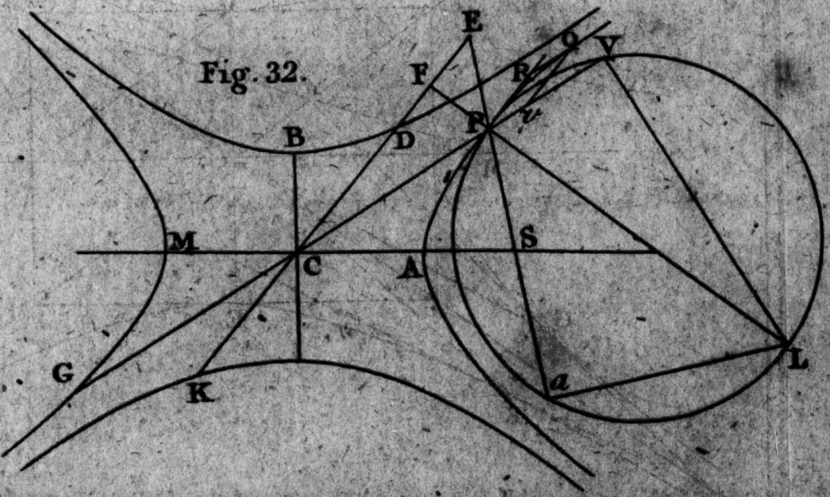
Fig. 32.